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THE DEVELOPMENT OF SUFFICIENT CONDITIONS IN THE CALCULUS OF VARIATIONS

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

By
WILLIAM LARKIN DUREN, JR.

L. C.

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This dissertation is one of a group of five submitted during the year 1930 to the Department of Mathematics of the University of Chicago and concerned with topics in the theory of the Calculus of Variations. The group as a whole has been published in book form by the University of Chicago Press under the title *Contributions to the Calculus of Variations, 1930.*

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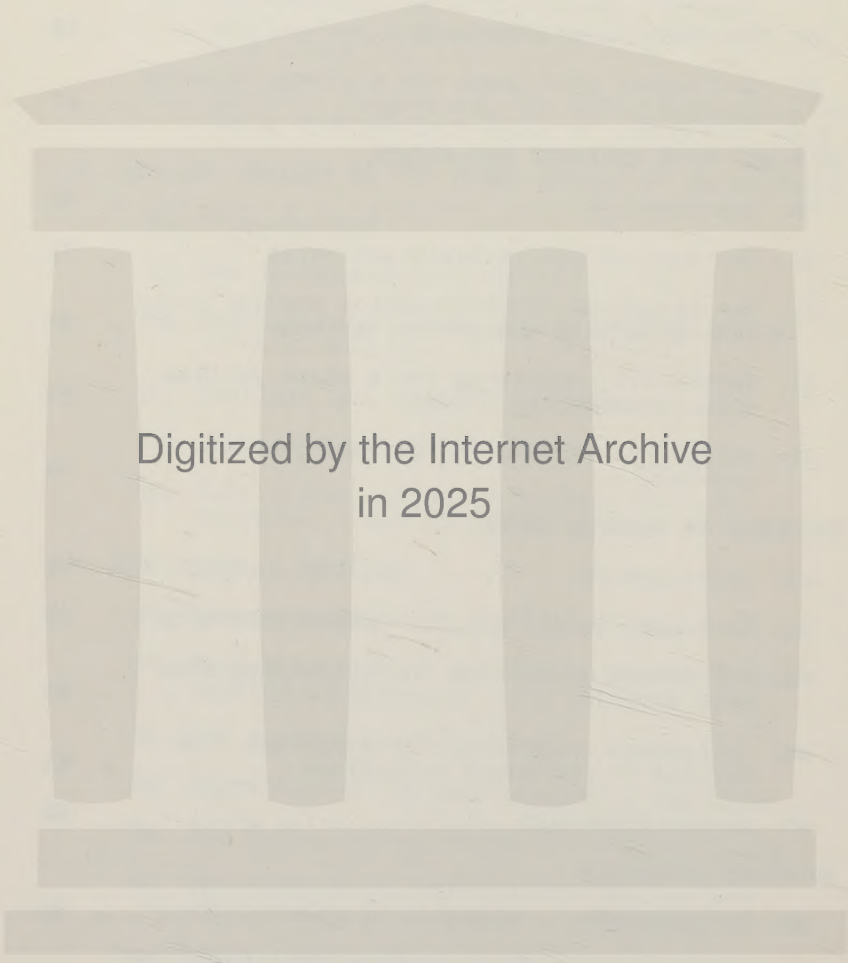

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INTRODUCTION

Although the calculus of variations has been a productive field of research in the last half century, comparatively few treatises have been written which give a comprehensive view of all phases of the subject. Much of the literature remains in scattered periodicals. It is the aim of this dissertation to present a critical and historical discussion of the results of those researches which had as their purpose the proof of the sufficiency of certain sets of conditions to insure the minimizing property. The growth of the consciousness of the desirability of sufficiency proofs will be traced and the proofs themselves analysed for various types of problems in the calculus of variations involving multiple as well as simple integrals.

The article on the calculus of variations by M. Lecat in the Encyclopedie des Sciences Mathematiques, II.31, covers the field briefly but with some completeness down to 1913, and the bibliographies of the same author published in 1913, 1923, and 1925 provide a list of papers which appeared up to 1925. Among the treatises Bolza's Vorlesungen über Variationsrechnung (1909) is a particularly useful historical and bibliographical source. The works of Lecat and Bolza and the Jahrbuch der Fortschritte der Mathematik have been perhaps the most important bibliographical guides in the preparation of this paper. The author has been able to consult original sources in his

study in all but a very few of the papers which contained appropriate material.

It is not proposed to describe all of the papers which have a bearing upon sufficiency proofs. Only the results of those which seem to be most important are to be presented in detail. The first chapter deals with the preliminary period from Euler to Weierstrass during which the comprehension of the importance of sufficiency proofs was gradually being developed. The first sufficiency proof was made by Weierstrass but the first was published by Scheeffer in 1886. These proofs were for a weak relative minimum and were made by expansion methods which form the subject of Chapter II. Chapters III and IV deal with the field method of Weierstrass as applied to sufficiency theorems for strong relative minima together with modifications and extensions of this method made by Kneser, Hilbert, Osgood, Carathéodory, Bolza, and Tonelli. Chapters V and VI are concerned with more general problems such as the Lagrange and Mayer problems and isoperimetric problems. The research of Mayer, Kneser, Bliss, Hahn, and Morse on sufficiency theorems for problems with variable end points is the subject of Chapter VII. Chapter VIII is concerned with special types of problems, discontinuous solutions, unilateral variations, and closed extremals. Chapter IX deals with multiple integrals.

The arrangement of the material will be according to the <u>method</u> of proof employed. This scheme also gives an approx-

imate classification by subject matter. The bibliography at the end of the paper is referred to by numbers in square brackets in the text.

CHAPTER I

THE CALCULUS OF VARIATIONS BEFORE THE TIME OF WEIERSTRASS

1. Introduction. The simplest problem of the calculus of variations in the plane is that of finding an arc which minimizes an integral of the form

$$I = \int_{x_1}^{x_2} f(x, y, y')\,dx \tag{1}$$

in a class of admissible arcs $[C_{12}]$ joining two given points, 1 and 2, and having the form

$$y = y(x), \qquad (x_1 \leqq x \leqq x_2). \tag{2}$$

The class of admissible arcs may be defined in a variety of ways but in every case it must be a class for each of whose arcs the integral I has a well defined value.

Early writers on the calculus of variations developed necessary conditions for a minimum but they did not have any theory of sets of conditions which could be proved to insure the minimizing property. The purpose of this dissertation is to follow through the literature of the calculus of variations the development of the nature and of the proofs of the effectiveness of certain sets of properties in insuring the minimizing property of an arc. The writers whose work is discussed in this first chapter are those of the period from Euler, who derived in 1744 the first important necessary condition for a minimum, to Weierstrass who made in 1879 the first adequate sufficiency proof. The work of Weierstrass himself will be discussed in Chapters II and III.

In order to discuss in more detail the work of the various writers it will be necessary first to introduce some of the more elementary notations of the calculus of variations theory. Let us focus attention upon a particular arc E_{12} of the class $[C_{12}]$ with the equation (2). We compare the value of the integral I taken over the arc E_{12} with its value taken over a comparison arc C_{12} of the class $[C_{12}]$ whose equation is

$$y = y(x) + \eta(x), \qquad (x_1 \leqq x \leqq x_2). \tag{3}$$

The total variation of the value of the integral I along E_{12} for the comparison arc C_{12} is the difference,

$$\Delta I = I(C_{12}) - I(E_{12}). \tag{4}$$

To get an expression for the difference (4) we substitute the expression (3) in the integrand of (1) and expand the integrand of the difference ΔI in powers of η by Taylor's formula. It is found that

$$\Delta I = \delta I + \delta^2 I + \cdots, \tag{5}$$

where δI, the so called first variation, is given by the equation

$$\delta I = \int_{x_1}^{x_2} (f_y \eta + f_{y'} \eta')\, dx, \tag{6}$$

and $\delta^2 I$, the second variation, is

$$\delta^2 I = \tfrac{1}{2} \int_{x_1}^{x_2} (f_{yy} \eta^2 + 2 f_{yy'} \eta \eta' + f_{y'y'} \eta'^2)\, dx. \tag{7}$$

The arguments of the derivatives of f are the x, $y(x)$, $y'(x)$ belonging to the arc (2). The first variation δI of the integral along a minimizing arc must vanish. In 1744 Euler derived

from this fact his famous differential equation condition [12]* which says that the function $y(x)$ defining a minimizing (or maximizing) arc satisfies the differential equation

$$f_y - \frac{d}{dx} f_{y'} = 0. \tag{8}$$

We will give the name extremal to an arc of the class $[C_{12}]$ which satisfies equation (8) and has continuous first and second derivatives. This definition of an extremal was not made by the earlier writers. It belongs to the more modern terminology.

It will be seen in later paragraphs that the writers of the period discussed in this chapter felt some distrust of the common belief of their time that an extremal surely have the minimizing or maximizing property. However, they never definitely formulated the problem of finding conditions sufficient to insure that an arc gives to the integral I the least value among those given by the arcs of the class $[C_{12}]$. It remained for Weierstrass to give the first sufficiency proofs about 1879.

The foundations were laid for the sufficiency proofs of Weierstrass in the earlier period. In 1760 de Silvabelle found that an extremal arc did not always furnish either a maximum or a minimum for a special problem of the calculus of variations formulated by Newton. In 1786 Legendre published a criterion by means of which he proposed to "distinguish maxima from minima." From this has come the Legendre necessary condition which

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*The numbers in square brackets refer to the bibliography at the end of the text.

says that along a minimizing extremal arc the inequality

(9) $$f_{y'y'} \geqq 0$$

must hold. The inequality is reversed for a maximizing arc. Legendre's conclusions were based upon a transformation of the second variation which Lagrange in 1797 showed was not always possible. In 1837 Jacobi derived a condition which showed the conditions under which Legendre's transformation fails. This condition was later named Jacobi's necessary condition for a minimum. In 1855 Bonnet gave an almost complete proof that on a surface of negative curvature a geodesic arc E_{12} is the shortest arc in a class of arcs $[C_{12}]$ lying in a sufficiently small neighborhood of E_{12} and joining the two points. In 1861 Todhunter pointed out the limitations of the expansion method for investigating the minimizing property of extremals. No new contributions to the development of the theory of sufficiency conditions were published between this time and the year 1879 in which Weierstrass first gave in lectures a complete sufficiency proof.

2. The period from Euler to Jacobi. Soon after Euler published his necessary condition in 1744 writers began to doubt that an extremal arc always furnished a minimum or a maximum for the problem. In 1760, in a study of Newton's problem [6, p. 411], de Saint-Jacques de Silvabelle [13, p.638] found that the arc along which the first variation vanished was not always actually a minimizing arc for the problem. He said that the extremal furnished only a "relative minimum," not an

"absolute minimum" for the problem, but it is difficult to determine just what he meant by these terms [13, p.646]. De Silvabelle used the old method of infinitesimal circle arcs which is due to Newton himself.

The first general investigation with a view to determining conditions under which the extremal arc furnished a maximum or a minimum for the problem was published by Legendre in 1786 [14, p.7]. In the beginning of this memoir he said:

"In most problems of the calculus of variations the nature of the question indicates whether a maximum or a minimum is possible, and which of the two has been determined. But there are cases where this distinction is not so easy, and that is chiefly when the problem admits neither an absolute maximum nor an absolute minimum ....

"I have thought, therefore, that the determination of a criterion to distinguish maxima from minima would serve as a supplement to the calculus of variations ...."

In studying the minima of the function $f(x,y)$ one expands the difference, $f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) = \Delta f$, by Taylor's series and finds

$$\Delta f = (f_x \Delta x + f_y \Delta y) + 1/2\,(f_{xx}\Delta x^2 + 2 f_{xy}\Delta x \Delta y + f_{yy}\Delta y^2) + \cdots$$

Then if $f_x = 0$ and $f_y = 0$ at (x_0, y_0) , and if the inequality,

$$f_{xx}\Delta x^2 + 2 f_{xy}\Delta x \Delta y + f_{yy}\Delta y^2 > 0,$$

holds at (x_0, y_0) for $|\Delta x|$ and $|\Delta y|$ sufficiently small, the point (x_0, y_0) is a minimum point. Similarly Legendre thought

that if $\delta^2 I$ in the expansion (5) is positive the arc considered is a minimizing arc. He attempted to find conditions sufficient to insure that the form in (7)

$$P\eta^2 + 2Q\eta\eta' + R\eta'^2, \tag{10}$$

where $P=f_{yy}$, $Q=f_{yy'}$, $R=f_{y'y'}$, is positive definite. By methods which he did not altogether justify he put the second variation (7) in the form

$$\delta^2 I = \frac{1}{2}\int_{x_1}^{x_2} R\left(\eta' + \frac{Q+w}{R}\eta\right)^2 dx, \tag{11}$$

where w is a solution of the equation

$$\left(P + \frac{dw}{dx}\right)R - (Q+w)^2 = 0. \tag{12}$$

Therefore, he said that when $R>0$ the second variation is surely positive and the integral I is a minimum along the curve defined by (8) [14, p.10].

Legendre himself felt that the investigation was unsatisfactory. According to his own statement, he could not show that a solution of (12) exists. In 1797 Lagrange [15, p.208] pointed out that Legendre's transformation (11) fails if w becomes infinite between the limits of integration. He did not resolve the difficulty, however. Another criticism was due to Legendre himself [14, p.23]. He found in Newton's problem that an extremal could be found which satisfied, in addition, his criterion $R>0$. But he constructed another arc which gave the integral a smaller value than that given by the extremal arc. This was his famous zig-zag line example. He was at loss

to explain this paradox and conjectured that the minima found by the analysis were only "relative or accidental" [14, p.24].

The primary object of Jacobi's memoir of 1837 [19,p.82] was to answer Lagrange's objection. His transformation of the second variation was different from that of Legendre. The possibility of Jacobi's transformation depended upon the existence of a non-vanishing solution of Jacobi's equation which is obtainable from (12) by the transformation,

$$w = \left(Q + R\frac{u'}{u}\right).$$

A solution of Jacobi's equation exists which does not vanish between the limits of integration if the extremal arc E_{12} contains no point conjugate to the initial point. The results of this memoir led to the formulation of Jacobi's necessary condition in the modern theory [6, p.87].

The distinction between necessary conditions and sufficient conditions was by no means clear during this period. See for example the works of Dirksen [16, p.21], Ohm [16,p.136] and Poisson [18] before Jacobi, and Ostrogradsky [20], Delaunay [21, p.96] Stegmann [24, p.102], and others, after Jacobi. Jacobi deplored the confusion which existed about the

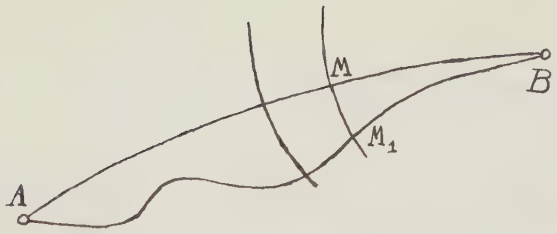
---

Jacobi said in his memoir 19, p.82
"Um auf das Maximum und Minimum zuruckzukommen: so ist ein Übelstand, dass im Gebrauch dieser Worter solche Verwirrung herrscht. Man sagt ein Ausdruck sei ein Maximum oder Minimum, wenn man nur sagen will, dass seine Variation verschwindet. Man sagt, eine Grosse sei ein Maximum, wenn man nur sagen will, sie sei kein Minimum."

meaning of the term, minimum, but he did not attempt to define the term himself. His 1837 memoir did not resolve Legendre's paradox of 1786 in Newton's problem and this remained an open question. In fact, when Jacobi stated without proof that his condition was necessary for a minimum, no less capable to mathematician than Bertrand [28, p.350] in 1855 criticized the statement. He conjectured that Jacobi had overstated his results in asserting that the condition was necessary for an extremal to be a minimizing arc. This is an interesting commentary upon the fact that contemporary mathematicians regarded Jacobi's result essentially as a sufficiency theorem.

3. The theory of geodesics. Gauss [27] in 1827 defined a geodesic curve on a surface as a curve whose osculating plane at every point contains the surface normal at the point. This definition is equivalent to the definition of a geodesic as an extremal for the shortest distance problem on the surface [6, p.211]. The question arises: is the geodesic always the shortest arc that can be drawn connecting two given points on a surface? In 1837 Jacobi [19, p.82] stated without proof that a geodesic is always the shortest arc on a surface whose total curvature is everywhere negative. Bonnet [29, p.32 ff.] undertook to prove this theorem in 1855 without the use of Jacobi's criterion which Bertrand had criticized. He also showed the necessity of Jacobi's condition for the problem.

Bonnet constructed a one-parameter family of geodesics orthogonal to the given geodesic , one through each point



M of AMB. The geodesics of this family intersect an arbitrary admissible arc AM_1B on the surface in points M_1. Let ω be the distance between the arcs, AMB and AM_1B, measured along the geodesic MM_1. s is the arc length of AMB measured from A and s_1 that of AM_1B. On a surface whose total curvature is everywhere negative, Bonnet found that to terms of the second order

$$\frac{ds_1}{ds} = \sqrt{\left(1 - \frac{\omega^2}{\rho_1\rho_2} + \cdots\right) + \omega'^2}, \tag{13}$$

where $1/\rho_1$ and $1/\rho_2$ are the principal curvatures of the surface at M_1. As a consequence of (13) he put the second variation in the form

$$\delta^2 I = \frac{1}{2}\int_0^s \left(\omega'^2 - \frac{\omega^2}{\rho_1\rho_2}\right) ds,$$

to terms of the second order in ω. Thus if $1/\rho_1\rho_2 < 0$ at every point of the surface, the second variation is surely positive for arcs in a sufficiently small neighborhood of the geodesic AMB. Bonnet concluded that the geodesic is the arc of least length joining A and B.

Though Bonnet seemed to have been unaware of it, he could have derived a stronger conclusion directly from (13). From (13) it follows that

$$\frac{ds_1}{ds} > 1$$

when $|\omega|$ is sufficiently small. Hence $\mathcal{S}_1 > \mathcal{S}$ and the geodesic gives a "strong relative minimum" for the problem. It will be seen in Chapter II that it can only be said that the extremal furnishes a "weak relative minimum" if $\delta^2 I$ in the expansion (5) is positive.

4. The state of the theory just prior to the time of Weierstrass. Before Weierstrass all attempted sufficiency proofs made use of the expansion (5). All writers made the tacit assumption that along an extremal the second variation, $\delta^2 I$, somehow dominated the higher order terms of the series. Therefore they said that "for small variations" ΔI had the same sign as $\delta^2 I$. Todhunter pointed out in 1861 [25, p.3], p.426 that the expansion (5) is valid only when both quantities $|\eta|$ and $|\eta'|$ are restricted. Although he realized the undesirability of the restriction upon the directions of comparison arcs, he did not inquire whether the condition could be removed. Furthermore he did not seem to think it necessary to show that $|\eta|$ and $|\eta'|$ could be chosen so small that ΔI had the same sign as $\delta^2 I$. He did apply his observation to indicate for the first time the reason for the failure of Legendre's analysis in the case of the zig-zag line example in Newton's problem. He showed that the zig-zag line did not satisfy the restriction upon the $|\eta'|$ [26, p.169].

Thus up to the time of Weierstrass's lectures at Berlin no complete sufficiency proofs were known, either for a weak relative minimum or a strong relative minimum. Writers had

not defined accurately the class of arcs in which a minimum was sought so that there was no rigorous basis for such a proof. However, there was a recognition of the need for sufficiency conditions. Arguments, such as Bonnet's, had been given which with moderate amplification were sufficient to insure a weak relative minimum, though no complete sufficiency proof had been given.

CHAPTER II

SUFFICIENT CONDITIONS BY EXPANSION METHODS

5. Introduction. The first sufficient conditions for a relative minimum were derived by Weierstrass about the year 1877. For sufficiency proofs he used two general methods, the expansion method and the field method. The former method is an extension of Legendre's idea of expanding the total variation into the form

$$\Delta I = I(C_{12}) - I(E_{12}) = \delta^2 I + \delta^3 I + \cdots, \tag{1}$$

as was done in equation (5) of Chapter I. This second chapter will be concerned with a study of sufficiency proofs based upon the expansion (1). The second method, which will be discussed in Chapter III, originated entirely with Weierstrass. It depends upon the notion of a field of extremals. Both of these methods became widely known to mathematicians through the lectures of Weierstrass in 1879 on the calculus of variations. These lectures were not published until 1927 when volume VII of the collected works of Weierstrass appeared, but students circulated copies of the lecture notes.

Two types of relative minima were first defined in these lectures. They may be described as follows. An arc E_{12} gives the integral I a weak relative minimum if there exists a neighborhood N in x, y, y'-space of the elements (x, y, y') on the arc E_{12} such that the value $I(E_{12})$ which E_{12} gives to the integral I is smaller than the value $I(C_{12})$ given by every other arc

C_{12} of the class $[C_{12}]$ whose elements lie in the neighborhood N. An arc E_{12} gives to I a strong relative minimum if there exists a neighborhood F of the points (x,y) of the arc E_{12} such that the arc E_{12} gives to the integral I a value smaller than that given by any other arc of the class $[C_{12}]$ whose points lie in the neighborhood F. The nomenclature, strong and weak relative minima, is due to Kneser in his Lehrbuch der Variationsrechnung published in 1900.

The first publication of a sufficiency proof for a general problem of the calculus of variations was that of L. Scheeffer in 1886. He attributed the ideas of this proof to Weierstrass. It is an expansion proof for a weak relative minimum. In a note in 1885 Scheefer [32, p.595] made a remark which helps to date the researches of Weierstrass. He said that he had last heard Weierstrass lecture in 1877, but that the report had come to him that Weierstrass had later succeeded in giving a sufficiency proof which made no restriction upon the derivatives of the comparison arcs, that is, a sufficiency proof for what would now be called a strong relative minimum. We know that Weierstrass gave such a proof in 1879, so the work must have been done between 1877 and 1879.

Numerous writers have given sufficiency proofs by the expansion method. Weierstrass studied the problem of the calculus of variations in the parametric form, which will be described in the first section of this chapter, and about 1877 he gave in his lectures an expansion proof of a sufficiency

theorem for a weak relative minimum for this case. Scheeffer gave his proof, as above mentioned in 1886. Kneser published a different proof in 1897. Other writers have since extended the expansion method to obtain sufficiency theorems for weak relative minima only. In 1894 Zermelo published the sufficiency proof of Weierstrass for a strong relative minimum of a simple integral using the notion of a field. The field method of Weierstrass was applied to minimal surfaces by Schwarz in 1885 and to problems involving double integrals with less accuracy by Kobb in 1892. The only published expansion proof which antedates the work of Zermelo is that of Scheefer in 1886. Since an arc which furnishes a strong relative minimum furnishes a fortiori a weak relative minimum, the proofs of Kneser and later writers by expansion methods are of interest only because of the method they used and not because of their results. In 1911 E. E. Levi published an interesting, though erroneous, attempt to obtain an expansion proof for a strong relative minimum. To date no such proof exists.

6. A simple problem of the calculus of variations in parametric form. Weierstrass [1] studied a simple problem of the calculus of variations in the plane in parametric form. This, is the problem of finding in a class $[C_{12}]$ of arcs of the form

$$x = x(t), \qquad y = y(t) \qquad (t_1 \leq t \leq t_2) \tag{2}$$

one which gives the integral

$$I = \int_{t_1}^{t_2} F(x, y, x', y')\,dt \tag{3}$$

its smallest value. The integrand function F is supposed to be positively homogeneous of the first order in x' and y', that is to say, the equation

$$F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y') \tag{4}$$

is satisfied for every $\kappa > 0$ and for $t_1 \leqq t \leqq t_2$.

Weierstrass formulated for this problem the three necessary conditions analogous to those of Euler, Legendre, and Jacobi and added a fourth necessary condition of his own which is called the Weierstrass necessary condition. The Euler condition says that along a minimizing arc without corners the differential equations

$$F_x - \frac{d}{dt} F_{x'} = 0, \quad F_y - \frac{d}{dt} F_{y'} = 0 \tag{5}$$

must be satisfied. Equations (5) may be shown equivalent to the single equation

$$F_{xy'} - F_{yx'} + F_1 (x'y'' - x''y') = 0 \tag{6}$$

The Legendre condition states that at every element (x, y, x', y') on a minimizing arc the inequality

$$F_1 (x, y, x', y') \geqq 0 \tag{7}$$

holds, where the function is defined by the equations

$$F_{x'x'} = y'^2 F_1, \quad F_{x'y'} = -x'y' F_1, \quad F_{y'y'} = x'^2 F_1. \tag{8}$$

The Jacobi necessary condition [1, p.153] states that a minimizing arc must contain in its interior no point conjugate to the initial point. The Weierstrass condition requires that at every element (x, y, p, q) on a minimizing arc E_{12} the inequality

(9) $$E(x, y, p, q, P, Q) \geqq 0$$

must hold for every set $(x, y, P, Q) \neq (x, y, p, q)$. In (9) the p and q are the direction cosines of the tangent to the arc E_{12} at the point (x, y), and where P and Q are arbitrary direction cosines. The E-function is defined by the equation

(10) $$E(x, y, x', y', X', Y') = F(x, y, X', Y') - X'F_{x'}(x, y, x', y') - Y'F_{y'}(x, y, x', y').$$

7. Convenient notations for necessary and for sufficient conditions. For the sake of convenience we will introduce short notations to stand for the conditions imposed upon an arc E_{12} in studying its minimizing property. In the parametric form of the problem it will be understood that the arc E_{12} is to be without multiple points. For the parametric problem the following conditions are imposed.

I. THE EULER CONDITION. The functions defining the arc E_{12} satisfy the differential equation (5) between the corners of E_{12}.

II. THE WEIERSTRASS CONDITION. Along the arc E_{12} the inequality (9) holds for admissible sets $(x, y, P, Q) \neq (x, y, p, q)$.

III. THE LEGENDRE CONDITION. The inequality (7) holds at each element (x, y, p, q) on the arc E_{12}.

IV. THE JACOBI CONDITION. The arc E_{12} contains in its interior no point 3 conjugate to the initial point 1.

We shall denote by II' the condition II strengthened to exclude the possibility of the equality in the inequality (9). Similarly III' excludes the equality in (7). The regularity

condition III'$_b$ requires that the inequality

(11) $$F_1(x, y, \cos\gamma, \sin\gamma) \geqq 0$$

holds for points (x,y) in a neighborhood of those of E_{12} and for arbitrary directions γ for which $(x, y, \cos\gamma, \sin\gamma)$ is an admissible set. The condition IV' demands all that is required in IV with the further restriction that the second end point is not conjugate to the initial point 1.

The same notations will be used for the analogous conditions in the non-parametric form of the problem, that of minimizing an integral of the form (1) of Chapter I. In condition I the Euler equation has the form (8) of Chapter I. In condition II the E-function is defined by the expression

$$E(x, y, y', Y') = f(x, y, Y') - f(x, y, y') - (Y' - y') f_{y'}(x, y, y').$$

A condition II'$_b$ is added which says that the inequality

$$E(x, y, y', Y') > 0$$

holds for sets (x, y, y') in a neighborhood of similar sets belonging to E_{12} and for admissible sets $(x, y, Y') \neq (x, y, y')$. In conditions III, III', and III'$_b$, the function F_1 is replaced by the function $f_{y'y'}(x, y, y')$.

These notations agree with those of Bliss [9, p.130]. Bolza [6, pp. 88, 116, 121] introduced notations of this type. His notations [illegible] dition I, Legendre condition II, Jacobi con[illegible] condition IV. Hereafter the conditions [illegible]red to by their numeral notations.

8. The first sufficiency theorem of Weierstrass. For the problem of minimizing the integral (3) among the values given it by a class $[C_{12}]$ of arcs joining two fixed points 1 and 2, Weierstrass [1, p.177] proved the following theorem.

THEOREM 1 OF WEIERSTRASS. *An arc E_{12} without corners and satisfying the conditions I, III', IV' furnishes a weak relative minimum for the integral in the parametric form (3) with respect to admissible arcs joining the points 1 and 2.*

By an admissible arc is meant an arc which is continuous and has a continuously turning tangent except possibly at a finite number of points called corners. At a corner on an admissible arc there exists a forward and a backward tangent. The character of an admissible arc is indicated by the figure above. The definition of these arcs was first published by Zermelo [2, p.25]. In the main the theory of the calculus of variations is concerned with admissible arcs as defined above, though Hilbert [43], Tonelli [8], and others have replaced the restriction upon the arcs as given above by less stringent ones.

Theorem 1 of Weierstrass remedied two significant deficiencies of the earlier writings. First it showed under what conditions the sign of the total variation coincides with that of the second variation when the first variation vanishes. Thus it supplied an answer to Todhunter's criticism on page 13

of Chapter I. Secondly it defined the class of arcs in which the arc E_{12} could be said to be the minimizing arc. A question about this had been raised by du Bois Reymond [30,p.289].

The proof of Weierstrass's theorem 1 depended upon a transformation of the second variation into the form

$$\delta^2 I = \frac{1}{2}\int_{t_1}^{t_2} F_1\left(w' - \frac{w u'}{u}\right)^2 dt, \tag{11}$$

where $w = x'\eta - y'\xi$ and the equations of a sample comparison arc C_{12} are

$$x = x(t) + \xi(t), \qquad y = y(t) + \eta(t), \tag{12}$$

and $u(t)$ is a non-vanishing solution of Jacobi's equation [1, p.153]. Weierstrass substituted the expressions (12) in the integrand of (3) and expanded the integrand to terms of the second degree in ξ, η, ξ', η' using Cauchy's integral form of the remainder for the second degree terms. He found that the total variation can be put in the form

$$\Delta I = \frac{1}{2}\int_{t_1}^{t_2} (F_1 - K)\left(w - \frac{w v'}{v}\right)^2 dt,$$

where K is an arbitrarily small positive constant and where is a non-vanishing solution of an equation of the same form as Jacobi's with small terms added to the coefficients. If K is chosen so small that the difference $F_1 - K$ is positive, theorem 1 of Weierstrass follows.

9. Scheeffer's method. The first published sufficiency proof was that of Scheeffer [33] in 1886. Scheeffer's theorem was the analogue for the non-parametric integral (1)

of Chapter I of theorem 1 of Weierstrass, the conditions I, III', and IV' being interpreted so as to apply to the non-parametric problem. The proof depended on a transformation of the second variation into the Jacobi form

$$\delta^2 I = \frac{1}{2}\int_{x_1}^{x_2} f_{y'y'}\left(\frac{\eta u' - \eta' u}{u}\right)^2 dx.$$

Scheeffer showed that there exists a constant a^2 such that the inequality

$$\delta^2 I > a^2 \int_{x_1}^{x_2} \zeta'^2\, dx, \tag{14}$$

where $\zeta = \eta/u$, always holds. He then expanded the total variation along E_{12} into the form,

$$\Delta I = \delta^2 I + R_3, \tag{15}$$

where R_3 is a remainder term of the third order in ζ and ζ'. By means of the integral inequality

$$\int_{x_1}^{x_2} \zeta^2\, dx \leqq \frac{(x_2 - x_1)^2}{\pi^2}\int_{x_1}^{x_2} \zeta'^2\, dx \tag{16}$$

he showed that the $|R_3|$ in (15) could be made smaller than the right member of (14). Hence he was able to conclude that ΔI was positive from (15). All later expansion proofs have followed the lines of either the proof of Scheeffer or that of Weierstrass.

10. The methods of Kneser and other writers. Kneser [34, p.95] pointed out in 1896 that the inequality (16) becomes useless and Scheeffer's method fails when the transformation of the second variation into the Legendre form (11) of Chapter I was employed instead of the form (13). By an argument like that

of Weierstrass he showed that the proof of Scheeffer's theorem could be made with the Legendre transformation. Bolza [4,p.69; 6, p.91] followed Kneser's proof. Hadamard [7, p.466] has given an account of both proofs.

In 1898 Kneser [35] gave a proof of an extension of Scheeffer's theorem to the problem of Lagrange. He made use of an elaborate theory of quadratic forms and the transformations of the second variation of Clebsch and A. Mayer. G. von Escherich [36] gave in 1902 a more elegant proof of the Kneser sufficiency theorem for a weak relative minimum for the problem of Lagrange. His proof made use of his own transformation of the second variation. He expanded the integrand to second order terms using the integral form of the remainder. G. A. Bliss [11] has given a proof somewhat like that of von Escherich of a sufficiency theorem for the three-space problem of the calculus of variations without side conditions. This proof depends upon a theorem also due to Bliss on differential equations containing arbitrary functions.

11. Levi's expansion proof of Lindeberg's theorem. Sufficient conditions for a strong relative minimum have been derived by the field method which will be the subject of the next chapter. For the simple problem with fixed end points in the non-parametric form the conditions I, II', III', IV' are not sufficient for a strong relative minimum as was shown by Bolza [37, p.8; 6, p. 116] in 1902. By use of the field method Lindeberg [38, p.534; 6, p.126] proved the following sufficiency

theorem for a semi-strong relative minimum.

LINDEBERG'S THEOREM 1. For the non-parametric problem an arc E_{12} without corners which satisfies the conditions I, II', III', IV' furnishes a semi-strong relative minimum for the problem with respect to admissible arcs joining the points 1 and 2.

By a semi-strong relative minimum it is meant that for every positive number ρ, however large, there exists a neighborhood $\mathcal{F}$ of the points of E_{12} such that the arc E_{12} gives to the integral a smaller value than that given by any other admissible arc joining the points 1 and 2, lying in $\mathcal{F}$, and having derivatives at each point whose values lie in a ρ-neighborhood of the set of values of the derivatives of .

E. E. Levi [39, p.425] in 1911 gave for Lindeberg's theorem an expansion proof which was an extension of the method of Scheeffer. He made use of the inequality (16). This theorem represents the strongest sufficiency theorem that has been obtained by expansion methods.

12. The failure of the expansion method for strong relative minima. It was the purpose of Levi [39, p.466] to prove by expansion methods the sufficiency of same set of conditions for a strong relative minimum. He regarded the notion of a field as an extraneous concept and one often attended with great analytical difficulties. He stated a sufficiency theorem for a strong relative minimum and attempted to give an expansion proof of it. Levi's work contained an error which H. Hahn [40] pointed out in 1913. In fact Levi's theorem was

false, as Hahn showed by an example. In a paper published in the same year as Hahn's paper Levi [41] himself disclosed the error and supplied an investigation which was intended as a proof by expansion methods of the sufficiency of the well known conditions [9, p.135] for a strong relative minimum in the non-parametric case, analogous to those of Weierstrass for parametric problems.

Though the theorem is correct, the work of Levi contains an error which seems to invalidate his proof. Levi's argument depended upon the use of the following lemma which is a variant of the inequality (16).

If $\zeta(x)$ is a function which is continuous and has a continuous derivative in the interval $x_1 \leqq x \leqq x_2$, and if $\zeta(x_1)=0$, then the inequality

$$(17) \qquad \int_{x_1}^{x_2} [\zeta(x)]^2 dx \leqq \frac{(x_2-x_1)^2}{2} \int_{x_1}^{x_2} [\zeta'(x)]^2 dx$$

holds.

It is easily seen that the lemma (17) does not hold in general if $\zeta(x)$ is discontinuous. For example it does not hold if $\zeta(x)$ is defined as follows:

$$\zeta(x)\equiv 0 \qquad \text{when} \quad x_1 \leqq x \leqq \tfrac{1}{2}(x_1+x_2),$$
$$\zeta(x)\equiv 1 \qquad \text{when} \quad \tfrac{1}{2}(x_1+x_2) \leqq x \leqq x_2.$$

Levi [41, p.189] has applied the inequality (17) with integrals taken in the sense of Lebesque. He denoted by χ the set of points in the interval (x_1, x_2) where $|\zeta'(x)| \leqq \rho$, ρ being an arbitrary fixed positive number. He denoted by χ_1 the complement of χ in (x_1, x_2). He defined functions ζ_1 and ζ_2

as follows:

$$\zeta_1 = \zeta(x) \text{ in } \chi, \qquad \zeta_1 \equiv 0 \text{ in } \chi_1,$$
$$\zeta_2 = \zeta(x) \text{ in } \chi, \qquad \zeta_2 \equiv \zeta(x) \text{ in } \chi_1,$$

so that $\zeta_1 + \zeta_2 = \zeta(x)$ in (x_1, x_2). Levi [41, p.186] has made the assumption that the inequality (17) holds for the function $\zeta_1(x)$. This use of the lemma is unjustifiable for the function $\zeta_1(x)$ can very well be discontinuous. The argument of Levi consequently fails. M. Malnate [42] in 1919 sought to extend the method of Levi to more general problems but this paper contains the same error as that of Levi. It seems, therefore, that no valid expansion proof of the sufficiency of the conditions I, II'$_b$, III', IV' for a strong relative minimum has yet been made.

CHAPTER III

THE FIELD METHOD OF WEIERSTRASS FOR SIMPLE PROBLEMS

13. Introduction. When Weierstrass introduced into the calculus of variations about 1879 the concept of a field of extremals it enabled him to make a proof of a sufficiency theorem for a strong relative minimum. Mathematicians who have later made sufficiency proofs for different and more general problems have followed the essential steps of his method. As we have seen from Scheeffer's remark mentioned on page 16, Weierstrass had derived sufficient conditions for a weak relative minimum, but he sought to eliminate the restriction upon the direction of comparison arcs. In this he was successful only after the derivation of a fourth necessary condition, the condition II of page 19, and the introduction of the concept of a field.

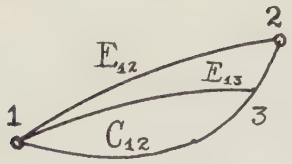
The field theory leading to sufficient conditions for a relative minimum for problems involving a simple integral was first published by Zermelo in 1894. His investigation followed closely the methods of Weierstrass. Kneser in 1900 published the theory with some valuable additions of his own. In the same year Hilbert introduced his invariant integral theory which greatly simplified the sufficiency proofs. W. F. Osgood in 1900 and others soon after gave accounts of sufficiency conditions based upon the work of Weierstrass.

This chapter will give an account of the sufficiency proofs mentioned in the preceding paragraph. It will be con-

cerned with sufficient conditions for plane problems involving a simple integral and having fixed end points. Also some results derived from these sufficiency proofs will be mentioned. These results are concerned with the existence of a minimizing extremal joining two points sufficiently near to each other, first proved in detail by Bliss in 1904; and further with a theorem of Osgood by means of which Osgood showed in 1901 that the sufficiency theorem holds for a more general class of comparison arcs than the class of admissible arcs considered by Weierstrass.

14. The field concept and the sufficiency proof of Weierstrass. Weierstrass showed that if an arc E_{12}, without double points or corners, satisfies the conditions I, III', IV' of page 19 then there exists a neighborhood F of the points (x, y) of E_{12} such that each point of F may be joined to the point 1 by one and only one extremal arc. Such a region F is known as a field [1, p.201]. Weierstrass himself used the term Flächenstreifens. The name Feld was introduced by Kneser [3, p.43]. Later writers have made refinements and expensions of the definition of a field by Weierstrass but the most of their sufficiency proofs depend fundamentally upon his ideas.

The original method of Weierstrass gives a clear idea of the usefulness of the field concept and the origin of the E-function. With Weierstrass we suppose that the extremal arc



is imbedded in a field F. We denote by any admissible arc lying in F and joining 1 and 2. An element on the arc C_{12} is (x, y, P, Q). Weierstrass calculated the derivative of the function

$$\phi(S) = I(E_{13}) + I(C_{32}) - I(E_{12}) \tag{1}$$

where S is the arc length of the arc C_{13} measured from the point 1 and where E_{13} is the extremal of the field through the point 3. He found this derivative to be given by the expression

$$\frac{d\phi}{ds} = -E(x_3, y_3, p_3, q_3, P_3, Q_3) \tag{2}$$

where the arguments (x_3, y_3, p_3, q_3) belong to E_{13} at the point 3, and P_3, Q_3 belong to C_{12} at the point 3.

Equation (2) is a form of the Weierstrass fundamental formula. By means of it he established the following theorem [1, p.224].

FUNDAMENTAL SUFFICIENCY THEOREM OF WEIERSTRASS. If the condition II' is satisfied at every point of a field F then the extremal arc E_{12} furnishes a minimum for the parametric problem with respect to admissible arcs joining 1 and 2 and lying interior to F.

In fact it is seen from equation (1) that

$$\phi(0) = I(C_{12}) - I(E_{12}), \qquad \phi(l) = 0$$

where l is the length of the arc C_{12}. Then, since the condition II' says that $E > 0$ for every set (x, y, p, q) of the field F and for admissible sets $(x, y, P, Q) \neq (x, y, p, q)$, formula (2) shows at once that the inequality

$$I(C_{12}) > I(E_{12})$$

is true. The theorem is thus established.

As was said on page 15, Weierstrass did not publish this proof. It was first published for a general problem involving a simple integral by Zermelo [2] in 1894. The problem studied by Zermelo was more general than that of Weierstrass. For Zermelo sought an arc which minimized an integral of the form

$$\int_{t_1}^{t_2} F(x, y, x', \cdots, x^{(n)}, y', \cdots, y^{(n)})\, dt$$

which contains higher derivatives in the arguments of the integrand. By a field method like that of Weierstrass, described in the preceding paragraph, he proved a sufficiency theorem with respect to arcs whose sets of values $(x, y, x', \cdots, x^{(n)}, y', \cdots, y^{(n)})$ lie in a neighborhood of those of the minimizing arc . For the problem of Weierstrass, when $n = 1$, this corresponds to a weak relative minimum. He also proved a sufficiency theorem relative to arcs whose values $(x, y, x', \cdots, x^{(n-1)}; y', \cdots, y^{(n-1)})$ lie in a neighborhood of those of E_{12}, and this corresponds to a strong relative minimum when $n = 1$.

15. The contributions of Kneser. The publication of Kneser's Lehrbuch [3] in 1900 made widely known the Weierstrass theory in complete form together with numerous contributions of Kneser himself. Among the things he did was to give the definition of a field a form more general and more desirable than that of Weierstrass. Kneser said that every one-parameter

family of extremals,

$$x = x(t, a), \quad y = y(t, a) \qquad (t_1 \leqq t \leqq t_2)$$

containing the arc E_{12} for a special value $a = a_0$ of the parameter forms a field if it simply covers a region F having all the points of E_{12} as interior points [3, p.43]. In plane problems the extremals of such a field are always cut transversely [3, p.47] by an admissible curve called a transversal of the field.

According to this definition, the family of extremals of Weierstrass described on page 29 does not form a field because an infinity of extremals pass through the point 1 which is a singular point of the field. The improper field employed by Weierstrass placed an undesirable restriction on the directions of comparison arcs at the initial point 1. To avoid this difficulty Kneser showed that if the arc E_{12} satisfies the conditions I, III', IV' it is possible to choose a point 0 on extended so near to the point 1 that the arc E_{02} contains no point conjugate to 0 [3, p.59]. The construction of a field was made with all the extremals passing through the point 0. When such a field has been shown to exist it is said that the Weierstrass construction is possible.

Kneser proved that in a field the formula

$$(3) \qquad I(C_{12}) - I(E_{12}) = \int_{t_1}^{t_2} E(x, y, p, q, P, Q)\, dt$$

holds, where the arguments (x, y, P, Q) belong to the arc C_{12} and the functions $p(x,y)$ and $q(x,y)$ are the direction cosines of the field extremal at the point (x, y). The formula (3) is

equivalent to the formula (2) and from it the fundamental sufficiency theorem of Weierstrass may be proved easily. Kneser proved the formula (3) by means of a differentiation under the integral sign with respect to the field parameter a.

Kneser extended the formula (3) to the case where the second end point is variable as will be discussed in Chapter IV. He also proved the analogue of the fundamental sufficiency theorem for the non-parametric problem [3, p.81]. Other contributions of his book were the envelope theorem, the Darboux-Kneser theory of curvilinear coordinates, and sufficiency theorem for more general problems.

16. Hilbert's invariant integral. In 1900 Hilbert [44, p.294] gave an elegant proof of the analogue of the fundamental formula of Weierstrass (3) for the non-parametric problem. Hilbert asked the question: Is is possible to determine the function $p(x,y)$ so that the integral

$$(4) \qquad I^* = \int_{x_1}^{x_2} \left\{ f(x,y,p) - (y'-p) f_{y'}(x,y,p) \right\} dx$$

shall be independent of the path in a region of (x,y) points? He found that if $p(x,y)$ is so determined then the solutions of the differential equation

$$\frac{dy}{dx} = p(x,y)$$

are extremals of a field. Conversely if $p(x,y)$ is the slope function of a field F then the integral I^* is independent of the path in F. The properties of the Hilbert invariant integral (4) are expressed by the relations

$$I^*(C_{12}) = I^*(E_{12}) = I(E_{12}) \tag{5}$$

From the relations (5) Hilbert derived at once the analogue of the Weierstrass formula,

$$I(C_{12}) - I(E_{12}) = \int_{x_1}^{x_2} E(x, y, p, y')\,dx. \tag{6}$$

The proof of Weierstrass of the fundamental sufficiency theorem, that of Kneser, and that of Hilbert were incorporated in a single paper by Osgood [45] in 1901. Osgood treated all three methods for the non-parametric problem and gave a critical comparison of them. Soon after 1900 numerous other accounts were published. The writers of these accounts were E. R. Hedrick [48], N. Gernet [49], Bolza [4, p.84; 47], Hancock [5, 137], E. Goursat [51], and V. P. Ermakov [52].

17. Sufficient conditions for a strong relative minimum. The proofs of Zermelo, Kneser, and Hilbert were proofs of the fundamental sufficiency theorem of page 30 in a field. To find a complete set of analytical conditions which will insure that the arc E_{12} furnishes a strong relative minimum one must find analytical conditions which insure that it is possible to imbed the arc E_{12} in a field. Bolza [6, p.100; 4, p.94] shifted the emphasis to sufficient conditions for a relative minimum. He showed that for both the parametric and non-parametric problems the set of conditions I, III', IV' insures the possibility of the Weierstrass construction of a field. This allowed the following theorems to be stated.

THEOREM. If for the non-parametric problem the arc is without corners and satisfies the conditions, I, II'$_b$,

III', IV', then E_{12} furnishes at least a strong relative minimum with respect to admissible arcs joining the points 1 and 2.

THEOREM. If for the parametric problem the arc E_{12} is without corners or double points and satisfies the conditions, I, II', III', IV', then E_{12} furnishes at least a strong relative minimum with respect to admissible arcs joining the points 1 and 2.

It should be noticed that the condition II'$_b$ for the non-parametric problem is replaced by the weaker condition II' for the parametric problem. For the parametric problem the condition II' implies the condition II'$_b$, as Zermelo [2, p.60] showed. However, Bolza [47] showed by example that the conditions I II', III', IV' are not sufficient for a strong relative minimum for the non-parametric problem. In both forms of the problem the condition II'$_b$ or II' may be replaced by the regularity condition III'$_b$ which implies that the condition II'$_b$ holds.

18. The existence of a minimizing extremal joining two points sufficiently near to each other. For regular problems it is always possible to construct an improper field of extremals radiating from a point and covering a neighborhood of the point. A regular problem in a region N of points (x, y) is a problem for which the condition III'$_b$ holds at every point of the region N. Although used by Weierstrass in his lectures and by Hilbert [43] in 1899, the following theorem was first proved by Bliss [50] in 1904.

THEOREM. If the problem is in parametric form and regular in a closed region N, there exists a positive number ρ such that, if P_1 is any point of N and P_2 is a point interior to the circle with center at P_1 and radius ρ, P_1 may be joined to P_2 by a unique extremal arc. The extremal E_{12} gives the integral a smaller value than any other admissible arc C_{12} in the circle (P_1, ρ). Furthermore if the problem is positive definite, that is if the inequality

$$F(x, y, \cos\gamma, \sin\gamma) > 0$$

holds for every point (x, y) of N and for every direction γ, then the arc E_{12} gives the integral a value smaller than any admissible arc lying in N and joining the points P_1 and P_2.

19. Osgood's theorem and the sufficiency proof for more general classes of comparison arcs. The theorem of Osgood states a fundamental property of a strong relative minimum. Osgood [46] first proved the theorem in 1901 for regular problems. Hahn [53] generalized the theorem in 1906 and the account which follows states the theorem in Hahn's form. Bolza also has given an account of it [6, p.280].

THEOREM OF OSGOOD. The arc E_{12} for the parametric problem is without corners or double points and satisfies the conditions I, II', III', IV'. Also at the end points of E_{12} the inequalities

(7) $F_1(x_1, y_1, \cos\gamma, \sin\gamma) > 0, \quad F_1(x_2, y_2, \cos\gamma, \sin\gamma) > 0$

hold for every direction γ. Then there exists a neighborhood Γ of E_{12} in which E_{12} furnishes a strong relative minimum for the integral. Furthermore for every closed neighborhood N of

E_{12} interior to Γ there exists a positive constant ϵ_N such that the inequality

$$I(C_{12}) - I(E_{12}) > \epsilon_N \tag{8}$$

holds for every admissible arc C_{12} which joins 1 and 2 and lies in Γ but not wholly in the neighborhood N.

As the parametric problem of the calculus of variations was stated in Chapter II, page 17, we seek a minimizing arc in a class $[C_{12}]$ of admissible arcs. The class of admissible arcs has heretofore been limited to the class defined by Weierstrass and described on page 21 of Chapter II. It was a class of arcs and such that integral

$$I = \int_{t_1}^{t_2} F(x, y, x', y')\, dt \tag{9}$$

in the Riemann sense has a well defined value along every arc C_{12} of the class. A more general and more natural definition of admissible arcs would be the class of continuous rectifiable arcs joining 1 and 2, but the Riemann integral (9) is not always well defined on such arcs.

In his lectures Weierstrass [1, pp.230-234] proposed to remedy this defect by a generalized definition of the integral $I(C_{12})$. He considered a finite partition π of the continuous, rectifiable arc C_{12}. The value of the generalized integral $I(C_{12})$ he defined to be

$$\lim_{\text{norm } \pi = 0} \sum_i F(x_i, y_i, \Delta x_i, \Delta y_i) \tag{10}$$

if this limit exists. The class of arcs along which the generalized integral is well defined Bolza designated as the

class K. With a theory of Riemann integration it could not be proved that the class K was the class of rectifiable arcs. However, by means of Osgood's theorem, Bolza [4] showed in 1904 that an extremal arc E_{12} which satisfies the hypotheses of Osgood's theorem gives the integral $I(C_{12})$ a value smaller than any arc of class K joining 1 and 2.

It remained to be proved that the class K is the class of continuous, rectifiable arcs. By a still further generalized idea of an integral Osgood [46] showed in 1901 that this is true for regular problems. When the integral (9) is defined in the Lebesgue sense Tonelli [8, I, p.217] and others have shown that it is well defined on a continuous rectifiable curve. Furthermore by means of Lebesgue integration theory it can be proved [8, I, p.303] that the Weierstrass generalized integral (10) is defined and equal to the Lebesgue integral on every continuous, rectifiable rac, and the class K therefore coincides with the class of continuous, rectifiable arcs. This result permits us to make the following extension of the sufficiency theorem.

SUPPLEMENTARY SUFFICIENCY THEOREM. An arc E_{12}, which satisfies the hypotheses of Osgood's theorem furnishes a strong relative minimum for the parametric problem with respect to continuous, rectifiable arcs joining the points 1 and 2.

CHAPTER IV

METHODS RELATED TO THE FIELD METHOD OF WEIERSTRASS

20. Introduction. There are three other methods of proving sufficiency theorems which are all somewhat related to the field method of Weierstrass discussed in Chapter III. These three methods form the subject of this Chapter IV.

The Darboux-Kneser method had an origin independent of the researches of Weierstrass. In 1889 G. Darboux [54] proved independently that on an analytic surface a geodesic arc which contains no pair of conjugate points is the shortest of all arcs on the surface which join its end points and lie in a neighborhood of the geodesic arc. His proof was based upon that of Bonnet mentioned in Chapter I, page 11. Darboux employed the Gaussian curvilinear coordinates on the surface with parameters , referred to a net of curves one of whose families was a family of geodesics and the other the orthogonal family of geodesic parallels. In 1900 Kneser [3, p.49] adopted Darboux's idea of curvilinear coordinates, extended it to a problem of the calculus of variations with a general integrand, and united the theory with that of Weierstrass.

It was Kneser [3, pp.69-74] who first pointed out in 1900 the close connection between the field theory of Weierstrass as supplemented by himself (see page 31) and a generalization of the classical Hamilton-Jacobi theory of dynamics. Caratheodory in 1904 introduced the method of geodesic parallels which enabled him to proceed at once to the fundamental suffi-

ciency theorem of Weierstrass without a previous study of necessary conditions. His theory is founded directly upon the Hamilton-Jacobi theory.

The third part of this chapter will deal with the sufficiency results which can be derived from the existence theorems of Hilbert, published in 1899, and of L. Tonelli, published in 1913. While neither of these methods actually makes use of the field theory they employ families of extremals similar to that of Weierstrass to prove sufficiency conditions for a relative minimum.

21. The Darboux-Kneser theory. Kneser [3, p.49] extended the method of Darboux, mentioned in the preceding paragraph, by replacing the geodesic arcs by the extremals of a field for the parametric integral

$$I = \int_{t_1}^{t_2} F(x, y, x', y')\, dt, \tag{1}$$

and replacing the family of geodesic parallels by the family of transversals to these extremals. The extremals were the curves,

constant, and the transversals were the curves $u = \text{constant}$ of a parametric net. The coordinates (x, y) in the integrand of (1) are related to the curvilinear coordinates u and v by a transformation of the form

$$x = X(u, v), \qquad y = Y(u, v) \tag{2}$$

and the integral (1) transforms into an integral of the form

$$I = \int_{\tau_1}^{\tau_2} G(u, v, u', v')\, d\tau \tag{3}$$

The E-function is an absolute invariant under a transformation

of the form (2). For every field of extremals in the u, v-plane having inclination θ' at the point (u,v), the fundamental formula of Weierstrass takes the analogous form

$$(4) \qquad I(C_{12}) - I(E_{12}) = \int_{\tau_1}^{\tau_2} E_G(u, v, \cos\theta', \sin\theta', u', v')\, d\tau$$

where E_G is the E-function formed for the integrand of (3). In the case when the field is formed by the parametric curves, $v = \text{constant}$, it is easily seen that the simplification

$$\cos\theta' \equiv 1, \quad \sin\theta' \equiv 0,$$

enters and furthermore that $E_G = G - u'$. Thus by the consideration of formula (4) Kneser united the theory of Darboux with that of Weierstrass.

22. The Hamilton-Jacobi theory and the method of Caratheodory. It has long been known that a family of extremals for the integral

$$(5) \qquad I = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

can be constructed if a solution $W(x,y)$ of the Hamiltonian partial differential equation

$$(6) \qquad W_x + H(x, y, W_y) = 0$$

is known and $W(x,y)$ has suitable analytical properties [6, p.138]. The curves,

$$W(x,y) = \text{constant},$$

are the transversals of a field of extremals. Along an arc of a transversal curve the Hilbert integral I^* of page 33 has the value zero. The extremals of the field are the solutions of the

differential equation

$$\frac{dy}{dx} = H_y(x, y, W_y). \tag{7}$$

The connection between the classical Hamilton-Jacobi theory and the fields of Weierstrass, Kneser, and Hilbert was pointed out by Kneser [3, p.69] himself in 1900 and later by Zermelo and Hahn [56, p.627].

Carathéodory has given a method of proving a form of the fundamental formula of Weierstrass which is closely allied to the Hamilton-Jacobi theory. His method has the advantage of not requiring a preliminary study of necessary conditions but it does not avoid the idea of a field completely as the method might at first seem to indicate.

He studied a family of curves

$$W(x, y) = \lambda \tag{8}$$

The family (8) is said to be a family of <u>geodesically equidistant curves</u> for the integral

$$I = \int_{x_0}^{x} f(x, y, y')dx$$

if the rate of change of I along a curve $y = y(x)$ which cuts those of the family (8) is dependent only on λ. This rate of change has the value

$$\frac{dI}{dx} = \frac{f}{\Delta}$$

where

$$\Delta = W_y y' + W_x > 0$$

The direction in which the quantity f/Δ is a minimum is

called the direction of geodesic descent at a point. If E_{12} is a curve of geodesic descent then

$$I(E_{12}) = W(x_2, y_2) - W(x_1, y_1)$$

and the following formula holds in a region simply covered by a family of geodesically equidistant curves of the form (8):

$$I(C_{12}) - (W_2 - W_1) = \int_{x_1}^{x_2} E(x, y, p(x, y), y')\, dx$$

where $p(x,y)$ is the direction of geodesic descent at the point (x, y). This formula at once justifies the familiar fundamental sufficiency theorem of page 30. For the function $W(x,y)$ defining a family of geodesically equidistant curves satisfies the Hamiltonian partial differential equation (6).

The method of Carathéodory was first published by him in 1904 [55]. Since then he has revised and extended it [57; 58; 59], and Bolza also has given a short account of it [6, p.128].

23. Absolute and relative minima. In the study of existence theorems for a minimum it is essential to distinguish between an absolute minimum for the problem and a relative minimum such as was defined in Chapter II, page 15. If the integral $I(C_{12})$ is well defined for every arc C_{12} of a given class $[C_{12}]$ of arcs and if E_{12} is an arc of the class such that the inequality

$$I(C_{12}) - I(E_{12}) \geqq 0$$

holds for every arc C_{12} of the class $[C_{12}]$, then the arc E_{12} is said to furnish an absolute minimum with respect to the class $[C_{12}]$. This is Tonelli's definition [8, II, p.3]. It

is more general than Bolza's definition [6, p.14]. For a relative minimum as defined in Chapter II, §5, the last inequality is not required to hold in the whole class $[C_{12}]$ but only in the subclass of $[C_{12}]$ whose arcs lie in a sufficiently small neighborhood of E_{12}.

24. Hilbert's existence theorem. Hilbert first published his existence theorem in 1899 [60]. He did not give many details and these have been supplied by later writers, namely, by Noble [61] in 1900, by Lebesgue [62] in 1902, by Carathéodory [63] in 1906, by Bolza [4, p.245] in 1904, and by Bolza [6, pp.419-443] in 1909. This report is based upon the last named work.

For the Hilbert existence theorem the class $[C_{12}]$ is the class of all continuous, rectifiable arcs joining two given points in a region R_0. The integral (1) in parametric form is given a generalized definition so that it is defined along all continuous rectifiable curves provided the hypotheses of the following theorem are satisfied.

THEOREM OF HILBERT. *If the integrand function of the integral (1) satisfies the following conditions:*

(a) $F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y')$ *for every* $\kappa > 0$

(b) *the problem is positive regular in* R_0,

(c) *the problem is positive definite in* R_0,

(d) R_0 *is a bounded, closed, convex region,*

then for every two distinct points 1 *and* 2 *of* R_0 *there exists a rectifiable arc* H_{12}, *joining* 1 *and* 2, *which furnishes an absolute minimum for the integral* (1). *The arc* H_{12} *is com-*

posed of segments of extremal arcs and possibly portions of the boundary of R_0. At interior points of R_0 the arc H_{12} is an extremal.

The Hilbert existence theorem justifies the following sufficiency theorem, stated by Bolza [6, p.436].

THEOREM. If the hypothesis of Hilbert's theorem are satisfied in a region R_0 which is bounded, perfect, simply connected, and extremal convex [6, p.276], and if there exists a unique extremal E_{12} in R_0 joining 1 and 2, then E_{12} furnishes the absolute minimum for the problem in R_0.

25. Sufficient conditions dervied from the existence theorems of Tonelli. The method of Tonelli appeared in a series of articles from about 1911 but the theory is collected in his Calcolo delle variazioni [8, II] published in 1923. His existence theorems require less restrictive hypotheses than that of Hilbert. He made use of the theory of Lebesgue integration so that the integral (1) is defined along every continuous, rectifiable arc. His main existence theorem is the following.

THEOREM OF TONELLI. If $I(C)$ is quasi-regular, positive definite in a complete class of curves $[C]$ which are all contained in a bounded region of x,y-space, then there exists a curve of the class $[C]$ which gives to the parametric integral (1) its minimum value for arcs of the class $[C]$.

The integral is quasi-regular if the inequality

$$F_1(x, y, x', y') \geqq 0$$

holds for every set (x, y, x', y') belonging to an arc of the class [C] and positive definite if the inequality $F(x, y, x', y') > 0$ holds for the same values. Tonelli first obtained a generalization of Hilbert's existence theorem in 1913 [64]. His method of proof was somewhat different from that of Hilbert and nowhere made use of the field theory of Weierstrass.

Out of the methods of these researches Tonelli [8, II, p.581] constructed a new proof that an extremal arc satisfying the conditions I, II', III', IV' furnishes a strong relative minimum for the problem. This is the second theorem of page 35. Tonelli's proof depends upon the existence of a minimizing extremal arc joining two points sufficiently near to each other 8, II, pp.154-164 , which he proved as a corollary of his general existence theorem, and upon a generalization of a theorem of Lindeberg [8, I, p.320; 87].

While the proof of Tonelli made no use of the field theory, the Hilbert invariant integral, or the fundamental formula (3) of Chapter III, it did involve the construction of a one parameter family of extremals through the initial point and simply covering a region of points about E_{12}. Such a family forms an improper field so that the field concept has not been altogether avoided. Tonelli's methods are extensible to variable end point problems.

CHAPTER V

MORE GENERAL PROBLEMS

26. Introduction. The problem of Lagrange in the calculus of variations is that of finding in a class of arcs

$$y_i = y_i(x) \qquad (x_1 \leqq x \leqq x_2; i = 1 \cdots n)$$

which join two points 1 and 2 and satisfy a set of independent differential equations of the form

(1) $$\phi_\alpha(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \quad (\alpha = 1 \cdots m < n)$$

one which minimizes an integral

(2) $$I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\, dx .$$

A more precise formulation of this problem may be found in the 1925 lectures of Bliss [75]. An admissible arc is one which has suitable continuity properties and satisfies the differential equation (1).

The problem of Lagrange in ordinary form and with fixed end points, as stated above, includes many special problems of the calculus of variations. It includes, for example, the simplest problem in the plane stated in the introduction of Chapter I. It includes the simple three-space problem which is that of minimizing an integral of the form (2) with $n = 2$ and with no side conditions of the form (1). Similarly it includes the space problem with no side conditions. The isoperimetric problems and other general problems can be restated in the form of a Lagrange problem. Although the proofs of sufficiency theorems for the three-space problem, the n-space

problem, and the isoperimetric problem were important preliminary steps in the development of the theory, nevertheless these proofs will not be discussed in detail here. The general sufficiency theory for the Lagrange problem with fixed end points, stated at the beginning of this section, has been developed and this chapter will be principally concerned with these proofs.

The principal difficulty in the way of the extension of the theory of Weierstrass, Kneser, and Hilbert to problems in higher spaces is that, although an n-parameter family of extremals of the integral (2) may simply cover a region of $x, y_1, \cdots, y_n$-space, this family may not furnish the slope functions with which the generalized Hilbert integral is independent of the path. A. Mayer in 1905 gave the first criterion for the existence of a field in which the invariantive property of the Hilbert integral holds. Such a field is called a Mayer field. Other criteria were deduced by Hilbert in 1905, Bolza in 1906, and Hahn in 1910. The analogue of the fundamental sufficiency theorem was first proved by Mayer in 1903. Analytic conditions sufficient for a relative minimum were first proved by Bolza in 1909. The proof of the sufficiency of these conditions was also made by Bliss in 1925, who also gave in 1924 another definition of a Mayer field which placed the emphasis upon the independence of the Hilbert integral rather than on a family of extremal arcs.

Another general problem in the calculus of variations is the problem of Mayer. Sufficient conditions for a strong relative minimum for this problem have not been derived but the

analogue of the fundamental sufficiency theorem in a field has been proved. This problem was studied by Kneser in 1900 and Egarov in 1906 but the first rigorous proof of the fundamental sufficiency theorem was that of G. A. Larew in 1924. Kneser has a proof of an analogous theorem for a parametric formulation of the problem published in 1925.

27. Necessary conditions for a minimum in the problem of Lagrange. The following statements of the necessary conditions for the problem of Lagrange are taken from the 1925 lectures of Bliss.

I. THE MULTIPLIER RULE [75, p.10] For every minimizing arc E_{12} without corners there exists a constant λ_0 and a set of continuous functions $\lambda_\alpha(x)$ such that the function

$$F(x, y, y', \lambda) = \lambda_0 f + \sum_{\alpha=1}^{m} \lambda_\alpha \phi_\alpha \tag{3}$$

satisfies the differential equations

$$\frac{d}{dx} F_{y_i'} = F_{y_i}, \qquad \phi_\alpha(x, y, y') = 0 \quad \begin{pmatrix} i = 1, \cdots, n \\ \alpha = 1, \cdots, m \end{pmatrix} \tag{4}$$

An admissible set of solutions

$$y_i = y_i(x), \ \lambda_0, \ \lambda_\alpha = \lambda_\alpha(x) \quad (x_1 \leqq x \leqq x_2) \tag{5}$$

of the Euler-Legrange equations (4) is called an extremal if it has continuous derivatives $y_i'(x)$, $y_i''(x)$, $\lambda_\alpha'(x)$. An extremal along which the determinant

$$R = \begin{vmatrix} F_{y_i' y_\kappa'} & \phi_{\beta y_i'} \\ \phi_{\alpha y_\kappa'} & 0 \end{vmatrix} \qquad \begin{pmatrix} \alpha, \beta = 1, \cdots, m \\ i, \kappa = 1, \cdots, n \end{pmatrix}$$

is different from zero is a member of a $2n$-parameter of extremals

(6) $y_i = y_i(x, a_1, \cdots, a_n, b_1, \cdots, b_n)$, $\lambda_\alpha = \lambda_\alpha(x, a_1, \cdots, a_n, b_1, \cdots, b_n)$

The extremals may be characterized by means of the canonical equations. The equations

(7) $$F_{y_i'}(x, y, y', \lambda) = v_i, \quad \phi_\alpha(x, y, y') = 0$$

may be solved for y' and λ as functions of x, y, v in the neighborhood of an extremal E_{12} along which $R \neq 0$. If these solutions are

(8) $$y_\kappa' = \Psi_\kappa(x, y, v), \quad \lambda_\beta = \Pi_\beta(x, y, v)$$

then the equations (4) may be replaced by the canonical [75, pp.12-15] differential equations

(9) $$\frac{dy_\kappa}{dx} = \Psi_\kappa(x, y, v), \quad \frac{dv_\kappa}{dx} F_{y_\kappa'}(x, y, \Psi, \Pi)$$

II. THE ANALOGUE OF THE WEIERSTRASS CONDITION [75,p.50]

At every element x, y, y', λ of a minimizing arc which is normal on every subinterval [75, p.49; 6, p.603] the inequality

(10) $$E(x, y, y', \lambda, Y') \geqq 0$$

must be satisfied for every admissible set $(x, yY') \neq (x, y, y')$.

The function E is defined by the formula

$$E(x, y, y', \lambda, Y') = F(x, y, Y', \lambda) - F(x, y, y', \lambda) + \sum_{i=1}^{n} (Y_i' - y_i') F_{y_i'}(x, y, y', \lambda).$$

III. THE CONDITION OF CLEBSCH [75, p.51] At every element (x, y, y', λ) of a minimizing arc which is normal on every sub-interval the inequality

$$\sum_{i=1}^{n} F_{y_i' y_\kappa'}(x, y, y', \lambda)\, \pi_i \pi_\kappa \geqq 0 \tag{11}$$

must be satisfied by every set $(\pi_1, \cdots, \pi_n) \neq (0, \cdots, 0)$ which is a solution of the m equations

$$\sum_{i=1}^{n} \phi_{\alpha y_i'}(x, y, y')\, \pi_i = 0$$

IV. THE CONDITION OF MAYER [75, p.62] A minimizing arc E_{12} can contain no point 3 conjugate [75, p.59] to 1 if E_{12} is an extremal normal on every sub-interval of (x_1, x_2) and if the determinant R is different from zero at every element of E_{12}.

28. The Hamilton-Jacobi theory. If we define the Hamiltonian function H by the equation

$$H(x, y, v) = \sum_{\kappa=1}^{n} \Psi_\kappa F_{y_\kappa'}(x, y, \Psi, \Pi) - F(x, y, \Psi, \Pi) \tag{12}$$

where the Ψ_κ and Π_β are defined in equation (8), then the canonical equation (9) may be written in the form [6, p.593],

$$\frac{dy_i}{dx} = H_{v_i}, \qquad \frac{dv_i}{dx} = -H_{y_i} \tag{13}$$

It is well known that if a solution $W(x, y_1, \cdots, y_n, a_1, \cdots, a_n)$ of the Hamilton-Jacobi partial differential equation

$$\frac{\partial W}{\partial x} - H\left(x, y_1, \cdots, y_n, \frac{\partial W}{\partial y_1}, \cdots, \frac{\partial W}{\partial y_n}\right) = 0 \tag{14}$$

is known with suitable analytical properties then the general solution of the system (13) is defined by the equations $\partial W/\partial a_\kappa = b_\kappa$ where the b_κ are arbitrary constants [6, p.600].

Mayer [66] showed in 1903 that if

(15) $y_i = y_i(x, a_1, \cdots, a_n)$, $\lambda_\alpha = \lambda_\alpha(x, a_1, \cdots, a_n)$

is an n-parameter family of extremals passing through the Kneser auxiliary point $(x_0, y_{10}, \cdots, y_{n0})$, described for the plane case in Chapter III, page 32, and found from a solution $W(x,y,a)$ of the Hamilton-Jacobi equation (14), then the equations

$$\frac{\partial}{\partial a_\kappa}\left[W(x,y,a) - W_0(x_0, y_0, a)\right] = 0$$

can be solved for $a_\kappa = a_\kappa(x,y)$. The functions

(16)
$$p_i(x,y) = \frac{\partial y_i(x,a)}{\partial x},$$
$$l_\alpha(x,y) = \lambda_\alpha(x,a)$$

where the functions $a_\kappa(x,y)$ are to be substituted for the constants a after the differentiation has been performed, associated with the extremals (15), are a set of slope functions and multipliers which furnish a solution of Hilbert's problem, page 33, for the problem of Lagrange. In other words the integral

(17) $$I^* = \int f(x,y,p)\,dx + \sum_{i=1}^{n} (dy_i - p_i\,dx) F_{y_i'}(x,y,p,l),$$

formed with the functions (16), is independent of the path in a region simply covered by the family (15).

29. Mayer fields for the problem of Lagrange. In a plane problem of the calculus of variations Kneser, page 31, had shown in 1900 that every one-parameter family of extremals which simply covers a region of x, y-space is cut by a transversal curve. Hilbert showed that such a family furnishes a slope function with which the integral I^* is independent of the path. In the first studies of the sufficiency theory for the problem of Lagrange writers said that every n-parameter family of extremal arcs

$$(18) \qquad y_i = y_i(x, a_1, \cdots, a_n), \qquad \lambda_\alpha = \lambda_\alpha(x, a_1, \cdots, a_n)$$

which simply covered a region of $x, y_1, \cdots, y_n$ - space formed a field [69]. However, it was very soon seen that not every family (18) possessed a transversal hypersurface or furnished the slope functions and multipliers (16) with which the Hilbert integral (17) was independent of the path.

As mentioned in the preceding section, Mayer showed in 1903 that a family of extremals (15) found from a solution of the Hamilton-Jacobi equation (14) does have the property of making the integral I^* invariant. It remained to prove necessary and sufficient conditions upon an arbitrary n-parameter family of extremals (18), found from the general solution of the differential equations (4) or (13), in order that the family (18) should furnish suitable slope functions and multipliers (16). Mayer studied the canonical solutions [75,p.12]

$$y_i = y_i(x, x_0, a, b), \qquad v_i = v_i(x, x_0, a, b)$$

of the differential equations (13), which are characterized by the initial conditions

$$a_i = y_i(x_0, x_0, a, b), \quad b_i = v_i(x_0, x_0, a, b).$$

Suppose that the constants b_i are replaced by functions $B_i(a_1, \cdots, a_n)$. Then the canonical system becomes an n-parameter family of extremals of the form

$$\begin{aligned} y_i &= y_i(x, a_1, \cdots, a_n, B_1, \cdots, B_n), \\ \lambda_\alpha &= \lambda_\alpha(x, a_1, \cdots, a_n, B_1, \cdots, B_n) \end{aligned} \tag{19}$$

when λ_α has been determined from equations (8). Mayer [67] proved in 1905 the following theorem.

CRITERION OF MAYER. If an n-parameter family of extremals defined by equations of the form (19) simply covers a region Γ of $x, y_1, \cdots, y_n$-space, then a necessary and sufficient condition that this family defines a set of slope functions and multipliers (16) with which the Hilbert integral (17) is independent of the path is that there exists a function $A(a_1, \cdots, a_n)$ such that the functions B_κ in (19) have the form

$$B_\kappa = \frac{\partial A}{\partial a_\kappa}.$$

A region Γ which has the properties described in the theorem is called a Mayer field. Hilbert [68] gave in 1905 another necessary and sufficient condition for the existence of a Mayer field. His criterion may be stated geometrically as follows.

CRITERION OF HILBERT. If an n-parameter family of extremals (18) simply covers a region $\mathfrak{F}$, and if the family is intersected by a hypersurface in $\mathfrak{F}$, and if the Hilbert integral I^* formed with the associated slope functions and multipliers (16) is independent of the path in the hypersurface, then I^* is independent of the path in the whole region $\mathfrak{F}$ and the region $\mathfrak{F}$ is a Mayer field.

It is evident from Hilbert's criterion that if all of the extremals of (18) pass through a common point, then a region which they simply cover is a Mayer field. The common point is a degenerate hypersurface on which the integral is invariant. The theory was also formulated geometrically by Bolza [69] who in 1906 extended to the Lagrange problem Kneser's theory of transversals of a field. He started with an n-parameter family of extremals

(20) $$y_i = y_i(x, a_1, \cdots, a_n), \qquad v_i = v_i(x, a_1, \cdots, a_n)$$

which simply cover a region $\mathfrak{F}$ and thus define inverse functions

(21) $$a_i = a_i(x, y_1, \cdots, y_n).$$

He undertook to determine when there exists an n-space

(22) $$x = \xi(a_1, \cdots a_n), \quad y_i = y_i(\xi, a_1, \cdots, a_n) = \eta_i(a_1, \cdots, a_n)$$

such that the partial derivatives of the field function,

(23) $$W(x, y) = \int_{\xi(a_1, \cdots, a_n)}^{x} F(x, y(x, a), y'(x, a))\, dx,$$

in which the inverse functions (21) are substituted for a, have the form

$$W_x = -H(x, y, v(x, a(x, y))),$$

$$W_{y_i} = v_i(x, a(x, y))$$

where H is defined by the relation (12).

CRITERION OF BOLZA. A transversal n-space (22) will exist for the family (20) if and only if the family (20) forms a Mayer field.

The n-spaces, $W(x, y) = \text{constant}$, are generalizations of the Kneser transversals of the field.

Another necessary and sufficient condition for the existence of a Mayer field was derived by Hahn [70] in 1910.

CRITERION OF HAHN. In order for an n-parameter family of extremals (18) which simply covers a region F to form a Mayer field it is necessary and sufficient that the equations

$$\sum_{i=1}^{n} \frac{\partial y_i}{\partial a_\nu} \frac{\partial F_{y_i'}}{\partial a_\mu} - \frac{\partial y_i}{\partial a_\mu} \frac{\partial F_{y_i'}}{\partial a_\nu} \qquad (i, \mu, \nu = 1, \cdots, n) \tag{24}$$

hold, where the arguments of F are the x, y, y', λ defined by the equations (18).

In the language of von Escherich [65, p.1236] the equations (24) say that the quantities $\partial y_i/\partial a_\mu, \partial \lambda_\alpha/\partial a_\mu$ form a conjugate system of solutions of the accessory equations. In his notation [65] the equations (24) have the form

$$\Psi\left(\frac{\partial y}{\partial a_\mu}, \frac{\partial \lambda}{\partial a_\mu}; \frac{\partial y}{\partial a_\nu}, \frac{\partial \lambda}{\partial a_\nu}\right) = 0. \tag{25}$$

According to a theorem of Clebsch the quantity Ψ is constant along an extremal so that if (25) holds on a hypersurface intersecting the family (18), it holds throughout the field F. In 1924 Bliss [74] proved Hahn's criterion in a simpler manner. In 1911 Bolza [72] published a paper in which the four criteria mentioned in this section were proved from one general theorem.

30. The definition of Bliss for a Mayer field. The essential property of a Mayer field is the invariance of the Hilbert integral (17). Bliss [75, p.64] stated in 1924 a definition of a Mayer field which emphasizes this property.

DEFINITION OF A MAYER FIELD. A Mayer field is a region F of $x, y_1, \cdots, y_n$- space, containing only interior points, and having associated with it a set of functions $p_i(x,y), l_\alpha(x,y)$ with the following properties:
(a) they have continuous first partial derivatives in F;
(b) the sets $(x,y,p(x,y))$ defined by the points in F are all admissible;
(c) the integral (17) formed with them is independent of the path in F.

There are well known analytic conditions [75, p.65] which are necessary and sufficient that a line integral of the form (17) be independent of the path in the class of all admissible arcs joining two points in F. The property needed in a field is only that the integral (17) be independent of the path in the class of arcs satisfying the differential equations (1). The invariant property is not needed for arcs not satisfying these equations. Bolza [73] showed in 1912 that the

necessary and sufficient conditions for the invariance of the integral (17) are the same whether the side conditions (1) are imposed or not. J. Radon [71] in 1911 showed that Bolza's results hold only when conditions of normality are imposed.

31. The fundamental sufficiency theorem for the problem of Lagrange. Once the properties of a Mayer field had been established the fundamental sufficiency theorem in a field was an easy consequence. It may be stated as follows:

FUNDAMENTAL SUFFICIENCY THEOREM. If E_{12} is an extremal arc of a Mayer field, and if at each point of the field the condition

$$E(x, y, p(x, y), l(x, y), y') > 0$$

is satisfied for all admissible sets $(x, y, y') \neq (x, y, p)$, then the inequality $I(C_{12}) \geqq I(E_{12})$ holds for every admissible arc C_{12} in the field and joining the points 1 and 2, and $I(C_{12}) = I(E_{12})$ only when C_{12} coincides with E_{12}.

The proof of this theorem is an easy extension of that described on page 34 for the simple problem. It was published first for the Lagrange problem by Mayer [66] in 1903.

32. Sufficient conditions for a relative minimum. The proof of the sufficiency for a strong relative minimum of a set of conditions analogous to the set on page 34 for the simple problem was first undertaken by Bolza [6, p.635 ff.] in 1909. He supposed that the extremal arc E_{12} was a part of a slightly longer extremal arc $E_{1'2'}$ which is "normal on every subinterval" [6, p.603; 75, p.49]. Then he showed that if the arc E_{12} in addition satisfied the conditions III' and IV' the

Weierstrass construction is possible. The condition III' means that the inequality (11) holds with the equality excluded. The condition IV' states that the conjugate point in the Mayer condition IV of page 51 is excluded from the end point as well as the interior of the arc E_{12}. As in Chapter III, page 32, the Weierstrass construction is said to be possible if there exists a point 0 on E_{12} extended such that an n-parameter family of extremals through 0 simply covers a region about the points of E_{12}.

To prove the possibility of the Weierstrass construction it must be proved that if the Mayer determinant [75, p.62]

$$\Delta(x, x_1, a_0, b_0) = \begin{vmatrix} y_a(x, a_0, b_0) & y_b(x, a_0, b_0) \\ y_a(x_1, a_0, b_0) & y_b(x_1, a_0, b_0) \end{vmatrix}$$

is different from zero on the interval $x_1 \leqq x \leqq x_2$ then there exists a point (x_0, y_0) on $E_{1'2'}$ but not on E_{12} so near to 1 that $\Delta(x, x_0, a_0, b_0) \neq 0$ for $x_1 \leqq x \leqq x_2$, provided that the conditions III' and IV' are satisfied on E_{12} and E_{12} is part of an extremal arc $E_{1'2'}$ which is normal on every subinterval. Bolza's proof in 1909 [6, p.632] of this lemma depended upon some complicated researches of Von Escherich on the second variation. The method of proof of Bliss [75, p.71] is simpler. Hilbert's criterion of page 55 shows at once that when the Weierstrass construction is possible the region covered by the extremals is a Mayer field.

Bolza in 1909 first proved the following theorem giving sufficient conditions for a strong relative minimum.

THEOREM. *If an admissible arc E_{12} is without corners and satisfies the conditions* I, II'$_b$, III', IV'; *and if E_{12} is part of an extended extremal arc $E_{1'2'}$ which is normal on every subinterval; then there is a neighborhood of the points (x,y) of E_{12} such that the inequality $I(C_{12}) > I(E_{12})$ holds for every admissible arc C_{12}, not identical with E_{12}.*

The condition II'$_b$ means that the inequality (10) holds without the equality in a neighborhood of the values, x, y, y', λ belonging to E_{12}. The statement of this theorem is like that of Bliss [75, p.70] published in 1925 except for the assumption that E_{12} is part of an extended extremal arc which is normal on every subinterval. Bliss assumes only that E_{12} itself is normal on every subinterval but the stronger condition seems necessary to carry out the proof.

Sufficiency conditions for a weak relative minimum were first obtained by the expansion method, as was stated in Chapter II, page 24. Bliss [75, p.73], however, proved by the methods described in the present chapter that the conditions of the preceding theorem for a strong relative minimum without the condition II'$_b$ are sufficient for a weak relative minimum.

33. The problem of Mayer. The problem of Mayer in the calculus of variations is that of finding in a class of admissible arcs

$$y_i = y_i(x) \qquad (x_1 \leqq x \leqq x_2)$$

having the end values

$$y_{i1} = y_i(x_1), \qquad y_{r2} = y_r(x_2)$$
$$(i = 0, 1, \cdots, n; \; r = 1, \cdots, n)$$

and satisfying the differential equations

$$\Phi_\alpha(x, y, y') = 0 \qquad (\alpha = 0, 1, \cdots, m \; ; \; m < n)$$

that one which gives a minimum value to $y_0(x_2)$.

The multiplier rule for the problem of Mayer says that for every minimizing arc without corners there exist functions $\lambda_\alpha(x)$ with which the functions $y_i(x)$ satisfy the differential equations

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0 \tag{26}$$

where $F = \sum_{\alpha=0}^{m} \lambda_\alpha \Phi_\alpha$. The condition of Weierstrass (II), the condition of Clebsch (III), and the condition of Mayer (IV) have formulations similar to the statement of these conditions for the Lagrange problem.

The fundamental sufficiency theorem was first treated by Kneser [3, p.250] in 1900. Egorov [76] also studied this question in 1906. He did not give many details and the paper has the same limitations as Mayer's paper [66] of 1903. He proved the fundamental sufficiency theorem only in a field all of whose extremals pass through a fixed point. In fact Mayer pointed out himself in his supplementary paper [67, p.349] that his general criterion for the independence of the Hilbert integral was not extended to the Mayer problem by his own proof.

Larew [78] in 1924 defined a Mayer field for the problem by means of ana analogue of the Hilbert invariant integral

I^*. She proved the fundamental formula

$$(27)\quad Y_0(x_2) - y_0(x_2) = \int_{x_1}^{x_2} \frac{-E(x, Y, p(x,Y), l(x,Y), Y')\,dx}{F_{y'}(x, Y, p(x,Y))},$$

where $y_i = Y_i(x)$ are the equations of an arbitrary admissible arc C_{12} satisfying the end conditions, and where $p(x, y_1, \cdots, y_n)$ and $l(x, y_1, \cdots, y_n)$ are the slope functions and multipliers of a field. With the help of formula (27) she proved an analogue of the fundamental sufficiency theorem for the problem. Kneser [10, p.292] in 1925 also proved an analogue of the fundamental sufficiency theorem in a field. He formulated the problem parametrically. His necessary and sufficient conditions for the existence of a field and his fundamental formula have a novel form.

34. Three-space problems. The Lagrange problem in the non-parametric form stated on page 47does not include the following parametric problem in three-space. The problem is that of finding in a class of arcs of the form

$$(28)\quad x = x(t), \quad y = y(t), \quad z = z(t) \qquad (t_1 \leqq t \leqq t_2)$$

one which gives a minimum value to the integral

$$I = \int_{t_1}^{t_2} F(x, y, z, x', y', z')\,dt,$$

where F satisfies the homogeneity condition

$$F(x, y, z, Kx', Ky', Kz') = K\,F(x, y, z, x', y', z') \quad (K>0)$$

Sufficient conditions for this problem were first derived by Mason and Bliss [79] in 1908. In fact this paper preceded the publication by Bolza of sufficient conditions for a strong relative minimum for the Lagrange problem in the sense

of page 58. Mason and Bliss proved that the analogues of the conditions I, III'$_b$, IV' as stated in Chapter II, page 20, are sufficient for a strong relative minimum. It is not essential to assume that the problem be regular and it can easily be seen that the condition III'$_b$ may be replaced by the weaker assumption that the conditions II'$_b$ and III' hold. The extension of these sufficiency conditions to n-space can readily be made once the F_1 function has been suitably defined and the Jacobi condition established as was done by Bliss [82, 83] in 1914 and 1916. The extension of the sufficient conditions of Bliss and Mason to extremal arcs in n-space has never been published. However, the theorem concerning sufficient conditions for extremal arcs in n-space is a special case of the sufficiency theorem of L. M. Graves [84] on discontinuous solutions published in 1930.

The corresponding problem in non-parametric form is a special case of the problem stated on page 47 and the sufficiency theorems obtained for it are included in those already described on pages 58 and 60. The theory for this case has been carried through by Bliss [11].

CHAPTER VI

ISOPERIMETRIC PROBLEMS

35. Introduction. An isoperimetric problem in the calculus of variations which was studied by Weierstrass is that of finding in a class of admissible arcs of the form

(1) $$x = x(t), \quad y = y(t) \qquad (t_1 \leqq t \leqq t_2)$$

which join two points 1 and 2, and satisfy the isoperimetric condition

(2) $$I = \int_{t_1}^{t_2} G(x, y, x', y')dt = \text{constant},$$

one which minimizes an integral

(3) $$I = \int_{t_1}^{t_2} F(x, y, x', y')dt .$$

The analogue of the necessary condition of Euler says that there exists a constant λ such that the functions (1) defining a minimizing arc satisfy between corners the Euler differential equations

(4) $$H_x - \frac{d}{dt} H_{x'} = 0, \quad H_y - \frac{d}{dt} H_{y'} = 0,$$

where H is defined by the equation

(5) $$H(x, y, x', y', \lambda) = F + \lambda G.$$

Similarly the necessary conditions II, III, and IV for the simple parametric problem given in Chapter II, page 19, have analogous formulations for the isoperimetric problem with replaced by H .

To study this problem Weierstrass adjoined a third variable, z [1, p.273]. Consider an extremal arc E_{1z} which satisfies the conditions I, III', IV'. Then a two-parameter family of extremals

$$x = x(t, a, \lambda), \quad y = y(t, a, \lambda) \tag{6}$$

passes through the point 0. For a given arc (1) the variable z is defined by the equation

$$z = z(t) = \int_{t_0}^{t} G(x, y, x', y')\, dt,$$

where the functions (1) and their derivatives have been substituted for the arguments of G.

By methods similar to those which are effective on page 62 of the preceding chapter it can be shown that the arc E'_{1z} in x, y, z-space, corresponding to E_{1z}, furnishes a strong relative minimum for the auxiliary problem if the conditions I, II', III', IV' are satisfied along E'_{1z}. This was proved by Weierstrass [1, p.282] in 1879 by setting up a field F' in x, y, z-space. It enabled him to state the following sufficiency theorem [4, p.244].

THEOREM OF WEIERSTRASS. *If the arc E_{1z} is without corners or double points and satisfies the conditions* I, II', III', IV' *then there exists in x, y, z-space a neighborhood of the arc E'_{1z} corresponding to E_{1z} such that the arc furnishes a minimum for I with respect to all admissible arcs C_{1z} satisfying the condition* (2) *whose corresponding arcs C'_{1z} in the x, y, z-space lie in the neighborhood F' of E'_{1z}.*

The theory of Weierstrass fails to give a sufficiency theorem for an unrestricted strong relative minimum with respect to all arcs satisfying the isoperimetric condition and lying in a neighborhood $\mathcal{F}$ of E_{12}. The difficulty was that an arc C_{12} which lies in the neighborhood $\mathcal{F}$ does not necessarily have a corresponding arc E'_{12} in x,y,z-space which lies in $\mathcal{F}'$. This difficulty was emphasized by Bolza in 1904 [4; p.242]. It was Lindeberg who in 1909 proved a special theorem by means of which sufficient conditions for a strong relative minimum for the isoperimetric problem in a neighborhood $\mathcal{F}$ of E_{12}. This theorem was later generalized and extended by E. E. Levi [86] in 1914 and by Tonelli [87] in 1921.

36. The theorem of Lindeberg. The theorem proved by Lindeberg concerned the minimum of the simple integral (3) without side conditions. If the arc L_{12} is admissible and of class C'', and moreover satisfies the conditions II' and III' of Chapter II, page 19, then there exists a neighborhood N of L_{12} such that N is simply covered by the family of normals to L_{12}. To each point of an admissible arc C_{12} lying in N there corresponds one point of L_{12} on the same normal. The angle between tangents at corresponding points he denoted by $\omega(\tau)$. For an arbitrary positive number ϵ' the set of points where $|\omega(\tau)| > \epsilon'$ is composed of a finite set of open intervals whose total length we denote by $m_{\epsilon'}(C_{12})$. The theorem of Lindeberg 85 proved in 1909 is the following.

THEOREM OF LINDEBERG. *For every pair of positive numbers,* ϵ *and* ϵ', *and every arc* L_{12} *which is admissible, of class* C'', *and satisfies the conditions* II', III' *there exists a neighborhood* $\mathfrak{F}$ *in the* x,y*-plane such that* $I(C_{12}) > I(L_{12})$ *for every admissible arc* C_{12} *different from* L_{12} *in the neighborhood* $\mathfrak{F}$ *for which the inequality*

$$m_{\epsilon'}(C_{12}) > \epsilon \tag{7}$$

holds.

The importance of this theorem depends upon the following lemma.

LEMMA OF LINDEBERG. *If the extremal arc* E_{12} *for the isoperimetric problem satisfies the conditions* I, II', III', IV', *then to every neighborhood* $\mathfrak{F}'$ *of the arc* E'_{12} *there corresponds a neighborhood* $\mathfrak{F}$ *of* E_{12} *and a pair of positive numbers,* ϵ *and* ϵ', *such that every admissible arc lying in* $\mathfrak{F}$ *and satisfying the condition*

$$m_{\epsilon'}(C_{12}) \leqq \epsilon \tag{8}$$

has a corresponding arc C'_{12} *which lies within* $\mathfrak{F}'$.

This lemma and the preceding theorem applied to E_{12} instead of L_{12} are readily seen to justify the sufficiency theorem for strong relative minima.

37. *Sufficient conditions for a strong relative minimum.* A minimizing arc for the integral

$$\int_{t_1}^{t_2} (F + \lambda G)\, dt$$

is evidently also a minimizing arc for the integral (3).

In 1909 Lindeberg proved his sufficiency theorem by dividing the class of all admissible arcs of class C'' joining 1 and 2 into two sub-classes by means of the lemma of page 67. The Weierstrass theorem of page 65 was applied to those satisfying the inequality (8); and the Lindeberg theorem was applied to those satisfying the inequality (7). His sufficiency theorem is the following.

THEOREM. *An arc E_{12} without corners or double points which satisfies the conditions* I, II', III', IV' *and the isoperimetric condition* (2) *furnishes a strong relative minimum with respect to admissible arcs C_{12} which join the points* 1 *and* 2 *and which satisfy the isoperimetric condition* (2).

CHAPTER VII

PROBLEMS WITH VARIABLE END-POINTS

38. Introduction. A simple parametric problem of the calculus of variations with variable end-points is that of finding in a class of arcs of the form

$$x = x(t), \quad y = y(t) \qquad (t_1 \leqq t \leqq t_2) \tag{1}$$

whose end-points (x_1, y_1) and (x_2, y_2) lie on two curves, D_1 and D_2 ; of the form

$$\begin{aligned} D_1: &\qquad x_1 = \xi_1(u), \quad y_1 = \eta_1(u), \\ D_2: &\qquad x_2 = \xi_2(v), \quad y_2 = \eta_2(v), \end{aligned} \tag{2}$$

one which minimizes the integral

$$I = \int_{t_1}^{t_2} F(x, y, x', y')dt. \tag{3}$$

This chapter will be concerned with problems with variable end-points which are of the simple type described above and with more general problems with variable end-points such as those of Lagrange and Mayer.

Kneser in 1900 was the first to deduce sufficient conditions for a general variable end-point problem. He extended the field method of Weierstrass to the case when the second end-point is variable but the first is fixed. In 1904 Bliss studied the problem stated at the beginning of this section. He dervied an extension of the Jacobi condition which enabled him to establish sufficient conditions for a minimum. Problems of more general form were studied by Mason and Bliss in 1908, Bolza in

and Radon in 1910, but the results obtained were limited.

For a Lagrange problem with a general type of end-conditions Mayer in 1884 outlined a method of making sufficiency proofs. His method reduces the problem to a problem of the calculus of variations with fixed end-points together with the problem of minimizing an ordinary function of several variables. Mayer's method is incomplete, but with a lemma which Hahn proved in 1911, providing an analytical theory for sufficiency proofs for problems of Lagrange with variable end-points, it provides a satisfactory method.

An analogue of the Jacobi condition for a problem of Mayer with variable end-points was derived by T. F. Cope in 1927. Using a formulation of the analogue of the Jacobi condition similar to that of Cope, M. Morse and E. Currier have announced recently in 1930 proofs of sufficient conditions for very general types of Lagrange and Mayer problems.

For the problem with variable end-points, stated at the beginning of this section, further necessary conditions for a minimum are needed to supplement those for the fixed end-point problem stated in Chapter II, page 19.

TRANSVERSALITY CONDITIONS [6, p.303]. The necessary condition of transversality states that at the end-points of a minimizing arc the equations

$$
\begin{aligned}
&F_{x'}(x_1, y_1, x_1', y_1')\,\xi_1' + F_{y'}(x_1, y_1, x_1', y_1')\,\eta_1' = 0, \\
&F_{x'}(x_2, y_2, x_2', y_2')\,\xi_2' + F_{y'}(x_2, y_2, x_2', y_2')\,\eta_2' = 0
\end{aligned}
\tag{4}
$$

hold, where the ξ' and η' are the derivatives of the functions ξ and η defined by equations (2).

The necessary condition of Jacobi (III) is to be replaced by an analogous condition which was proved necessary for a minimum by Bliss 92 in 1904. Instead of the conjugate point this condition involves the critical points [92], 1' and 2', for the curves D_1 and D_2 respectively on the extremal passing through 1 and 2.

THE CONDITION OF BLISS. On an extremal, the arc E_{12} of which is a minimizing arc, the end points and the critical points for D_1 and D_2 must occur in the cyclical order 1 2 2' 1'.

Kneser [3, pp.89-92] had published earlier, in 1900, an analogue of the Jacobi condition for the case when only one end point is variable on a curve. He gave the definition of a critical point (Brennpunkt) for a curve D_2 on the extremal through the point 1 which is cut transversally by D_2.

39. The case of one variable end point. Kneser [3,p.77] in 1900 derived sufficient conditions for a strong relative minimum in the case when the curve D_1 reduces to a point and the second end point alone is variable. He proved that if the arc E_{12} is an extremal of a field F of extremals which are cut transversally by the curve D_2, then an analogue of the formula of Weierstrass,

$$I(C_{14}) - I(E_{12}) = \int_{t_1}^{t_2} E(x, y, p, q, x', y')\,dt,$$

holds for every arc C_{14} joining the point 1 to every point 4 on D_2 and lying entirely in the field $\mathbf{F}$. The arguments (x, y, x', y') of E belong to C_{14} while the functions, p and q, are the slope functions of the field. He [3, p.109] also demonstrated that a certain set of analytic conditions are sufficient to insure the existence of a field of the kind just described. The combination of these results justifies the following sufficiency theorem.

THEOREM. *If E_{12} is an arc without corners or double points which joins the point 1 to the curve D_2 and is cut transversally by D_2 at the point 2; if along E_{12} the conditions* I, II', III' *of Chapter II, page 19, are satisfied; if the arc E_{12} contains no critical point of D_2; and finally if $F(x_2, y_2, x_2', y_2') \neq 0$; then E_{12} furnishes a strong relative minimum to the integral (3) with respect to admissible arcs joining the point* 1 *to the curve D_2.*

The extension of this method to a parametric problem in space was made by Mason and Bliss [94] in 1908. They derived sufficient conditions for the problem of Chapter V, page 62, for the case when the second end point is allowed to vary on a curve. For an isoperimetric problem with a variable end point Bolza [6, p.523] in 1909 derived sufficient conditions for a minimum. Radon [95, p.1294] in 1910 proved similar results for the problem of minimizing an integral of the form,

$$\int_{S_0}^{S_1} \mathcal{F}(x, y, \theta, \chi)\, dS,$$

when one end element is allowed to vary on an "element curve." The variables θ, χ in the integrand represent the inclination of the tangent and the curvature of the curve along which the integral is taken.

40. The method of differentiation applied to problems with both end points variable. This section will be concerned with a Lagrange problem with variable end points. The statement of the problem is the same as that stated in Chapter V, page 47, except that instead of seeking a minimizing arc in a class of admissible arcs joining two given points the minimizing arc is sought in a class of arcs whose ends lie on two manifolds of the form

$$(5)\qquad \begin{aligned} M_1&: \quad x_1 = \xi_1(u_1, \cdots, u_p), \quad y_{i1} = \eta_{i1}(u_1, \cdots, u_p), \\ M_2&: \quad x_2 = \xi(v_1, \cdots, v_q), \quad y_{i2} = \eta_{i2}(v_1, \cdots, v_q). \end{aligned}$$

Under certain conditions, if a solution $E_{12}^{(0)}$ of the Euler-Lagrange equations (4) of Chapter V which joins two points 1 and 2 exists, then every point in a sufficiently small neighborhood of 1 can be joined to every point in a sufficiently small neighborhood of 2 by a unique extremal arc. The equations of these extremal arcs have the form

$$(6)\qquad y_i = y_i(x, u, v), \qquad \lambda_\alpha = \lambda_\alpha(x, u, v).$$

If the values of y_i from (6) and the values of x_1 and x_2 from the conditions (5) are substituted into the integrand and limits of the integral (2) of Chapter V, it becomes a function

of the $p+q$ variables u and v,

$$(7) \qquad I(u,v) = \int_{\xi_1(u)}^{\xi_2(v)} f(x, y(x,u,v))\,dx.$$

The expression (7) gives the value of I for values in a u, v – neighborhood of the values $(u_1^{(0)}, \cdots, u_p^{(0)}; v_1^{(0)}, \cdots, v_q^{(0)})$ defining the initial extremal $E_{12}^{(0)}$. Suppose there exists a neighborhood $\mathcal{F}$ of the initial extremal $E_{12}^{(0)}$ such that every extremal arc, whose end points are determined by a set (u,v) near those defining $E_{12}^{(0)}$, furnishes a minimum to the integral (2) of Chapter V with respect to admissible arcs joining its own end points and lying in $\mathcal{F}$. Then to solve the problem it only remains to show that the point $(u^{(0)}, v^{(0)})$ is a minimum point for the function $I(u,v)$ defined by (7).

This is the method which Bolza [6, p.302] calls the method of differentiation. The outlines of the method were first published by Mayer [89] in 1884. However, it can be regarded as complete only in connection with a lemma proved by Hahn [96] in 1911. Hahn's lemma in a form suitable for application here may be stated as follows.

LEMMA OF HAHN. If $E_{12}^{(0)}$ is an extremal arc defined by the set $(u^{(0)}, v^{(0)})$ and if $E_{12}^{(0)}$ satisfies the conditions for a strong relative minimum for the fixed end-point problem as stated in Chapter V, page 60; then there exists a neighborhood N of the values $(u^{(0)}, v^{(0)})$ and a neighborhood $\mathcal{F}$ of $E_{12}^{(0)}$ such that every extremal arc E_{12} defined by values

(u, v) in N furnishes a minimum with respect to admissible arcs joining its own end points and lying in F.

As a consequence of his lemma Hahn deduced the following theorem on variable end-point problems.

THEOREM OF HAHN. If the extremal arc $E_{12}^{(0)}$ furnishes a weak relative minimum for the problem of Lagrange with variable end points; and furthermore if the condition II'$_b$ of Chapter V, page 60, is satisfied along $E_{12}^{(0)}$; then $E_{12}^{(0)}$ furnishes a strong relative minimum for the integral of the problem.

Hahn applied this theorem to the resolution of a difficulty which arises in the proofs of Bliss, Bolza, and Radon mentioned on page 72. The difficulty lies in that the fields constructed have singular points along the end curve D_2. In order to construct a field filling the whole of space surrounding D_2 stronger assumptions must be made than are provided by the condition of Legendre when strengthened in the usual way for the purpose of proving sufficient conditions.

41. Sufficient conditions for a plane problem with variable end-points. An example of the method of differentiation is the sufficiency proof by Bliss 92 in 1904 for the problem stated on page 69. The condition of Bliss in the s stronger form will be said to hold if no two of the points 1 2 2' 1' mentioned in §1 coincide.

SUFFICIENCY THEOREM OF BLISS. If the arc E_{12} is an admissible arc for the parametric problem with variable end points in the plane; if E_{12} is without corners or double

points; if E_{12} is cut transversally by the end curves D_1 and D_2; if along E_{12} the conditions I, II', III' of Chapter II, page 19, are satisfied; if the Bliss condition is satisfied in the stronger form; and finally if the conditions

$$F(x_1, y_1, x_1', y_1') \neq 0, \qquad F(x_2, y_2, x_2', y_2') \neq 0$$

are satisfied; then E_{12} furnishes a strong relative minimum for the integral of the problem.

A proof of a similar theorem but with stronger hypotheses was made by Zermelo and Hahn [93] in 1904. Their proof threw the question back onto the problem with a single variable end-point.

42. Sufficient conditions for more general problems. Mayer's method of differentiation together with the lemma of Hahn, which were described on page 74, justify at once the following theorem which gives sufficient conditions for a relative minimum for the problem of Lagrange with variable end points stated on page 73.

THEOREM. If the arc E_{12} defined by the parameter values $(u^{(0)}, v^{(0)})$ satisfies the sufficient conditions for the fixed end point problem, stated in the theorem of Chapter V, page 60; and if the function $I(u,v)$ defined by (7) has a proper minimum at $(u^{(0)}, v^{(0)})$; then the arc E_{12} furnishes a strong relative minimum to the integral of the problem.

The vanishing of the first partial derivatives of $I(u,v)$ at $(u^{(0)}, v^{(0)})$ is expressible in terms of the transversality conditions. It remains to show further that the usual quadratic form

$$(8)\sum_{r,s=1}^{p} I_{u_r u_s}\delta u_r \delta u_s + 2\sum_{r=1}^{p}\sum_{s=1}^{q} I_{u_r v_s}\delta u_r \delta v_s + \sum_{r,s=1}^{q} I_{v_r v_s}\delta v_r \delta v_s$$

is positive definite at $(u^{(0)}, v^{(0)})$.

The condition on the quadratic form (8) takes the place of an analogue of the Jacobi condition for the problem with variable end points. As was seen in the preceding section, on a plane problem, Bliss interpreted the condition on the form (8) by means of his critical point condition. It is usual in sufficient conditions to make use of the Jacobi condition or an analogue of it. Cope [98] in 1927 formulated an analogue of the Jacobi condition for problem of Mayer with very general end conditions. This problem includes the problem stated on page 73. M. Morse [99] has recently (1930) announced a theorem giving sufficient conditions for a strong relative minimum for a problem of Lagrange with variable end points with a formulation of an analogue of the Jacobi condition like that of Cope. To bring the general theory to the same complete form that Bliss did the plane problem, see page 75, it remains to find for the condition of Cope or a similar condition a geometrical interpretation in terms of the curvatures and critical points of the manifolds M_1 and M_2 defined by equations (5). E. Currier [100] has announced in 1930 some results in this direction for a parametric problem in n-space without side conditions of the form (1) of Chapter V.

CHAPTER VIII

PROBLEMS OF SPECIAL TYPES

43. Introduction. The minimizing arcs with which the chapter are concerned are arcs with corners, arcs which contain a portion of the boundary of definition of the problem, and closed extremals. Heretofore it has always been assumed that the arc to be investigated for a minimum is an extremal arc whose elements lie interior to the region of definition of the problem. Sufficiency proofs may be made in the three other cases named above but each proof demands the construction of a special type of field.

Sufficient conditions for arcs with corners were discussed by Carathéodory in 1904, by Bolza in 1908 and 1909, and by Graves in 1930. Sufficient conditions for arcs containing part of the boundary for the problem were treated by Weierstrass in his lectures of 1879, Bliss in 1904, Bolza in 1909, Bliss and Underhill in 1914, and others. Sufficient condition for closed extremal curves or periodic problems are due to Poincaré in 1899, Hadamard in 1910, Radon in 1922, and Carathéodory in 1925.

44. Necessary conditions for broken extremals. The problem best adapted to the study of broken extremals, or extremaloids as they are called by Graves [106], is the parametric form of the problem stated in Chapter II, page 17. The integral to be minimized has the form

(328)

$$(1) \qquad I = \int_{t_1}^{t_2} F(x, y, x', y')dt$$

where F satisfies the usual homogeneity conditions. The necessary condition of Weierstrass-Erdmann [6, p.368] says that at corners of a minimizing arc the equations

$$(2) \qquad \begin{aligned} F_{x'}(x, y, x'^-, y'^-) &= F_{x'}(x, y, x'^+, y'^+), \\ F_{y'}(x, y, x'^-, y'^-) &= F_{y'}(x, y, x'^+, y'^+) \end{aligned}$$

are satisfied, where x'^-, y'^- denote the backward direction of the arc at the corner and x'^+, y'^+ denote the forward direction. Furthermore at a corner O of an extremaloid it is usual to assume that $\Omega_o \neq 0$, where

$$\Omega_o = p_o^- F_x(x_o, y_o, p_o^+, q_o^+) + q_o^- F_y(x_o, y_o, p_o^+, q_o^+)$$
$$- p_o^+ F_x(x_o, y_o, p_o^-, q_o^-) - q_o^+ F_y(x_o, y_o, p_o^-, q_o^-).$$

The function Ω_o is the corner function of Carathéodory [101] who first studied fields of extremaloids in 1904. Bolza [102] also studied such fields.

Consider an extremaloid E_{1oz} composed of two extremal arcs, E_{1o} and E_{oz} . Suppose E_{1o} is an extremal of a field in the ordinary sense. Then, if $\Omega_o \neq 0$, a one-parameter family of extremal arcs is determined which includes E_{oz} . Each member of this family intersects the corresponding member of the family to which E_{1o} belongs in a point lying on a corner curve K_o [101]. Thus a one parameter family of extremaloids satisfying the corner condition is constructed. By

means of an envelope theorem Carathéodory [101; 102; 103] defined conjugate points 1 and 1' on an extremaloid. Graves [106] defined conjugate points for a n-space parametric problem in n-space by means of the second variation in a manner similar to that of Bliss [105] for extremals.

45. Sufficient conditions for minimizing arcs with corners. Let $0'$ denote a point of the extremal arc E_{10} such that the family of extremals through $0'$ defines a corner curve K_0 tangent to E_{02} at the point 0. Then, according to Bolza [6, p.385], the following sufficiency theorem may be stated, in which θ_0^- and θ_0^+ are the inclinations of the tangents to E_{102} at the point 0, and 1' is conjugate to 1.

THEOREM. If E_{102} is an extremaloid without double points; if the corner condition (2) is satisfied at the corner 0; if $\Omega_0 \neq 0$ and $\sin(\theta_0^+ - \theta_0^-) \neq 0$; if the critical point condition holds that on the extremal E_{10} the points 0' 1 0 occur in that order and on the extremal E_{02} the points 0 2 1' occur in that order; finally if the conditions II' and III' are satisfied along the extremaloid E_{102}; then E_{102} furnishes a strong relative minimum to the integral of the problem.

The proof depends upon the extension of the Weierstrass fundamental formula (3) of Chapter III to hold for this field of broken extremals.

In his sufficient conditions Graves 106 replaces the critical point condition above by the condition that neither end point of the extremaloid E_{102} has a conjugate point according

to his definition on the extremaloid. The conditions II' and III' are replaced by the condition that each extremal arc is "positively strong" [106, p.9]. This means that the function E_1 defined by Bliss [104] for space problems satisfies the inequality

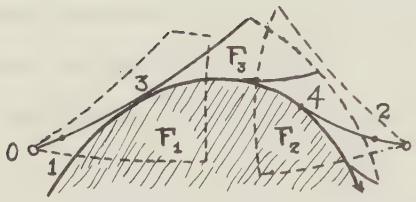
$$E_1(x, y, p, q, p_1, q_1, \omega) > 0$$

for every element (x, y, p, q) on the extremal and $0 \leqq \omega \leqq \pi$. The function E_1 is defined by the relation

$$E_1(x, y, p, q, p_1, q_1, \omega) = E(x, y, p, q, P, Q)/(1 - \cos\omega)$$

in which ω is the angle between the directions (p, q) and (P, Q) and (p_1, q_1) is a direction orthogonal to (p, q). Graves proved that the minimizing arc can have no cusp so that the condition $\sin(\theta_0^+ - \theta_0^-) \neq 0$ is not needed. He treated a parametric problem in n-space and his sufficiency proof was made for that problem. Also in his proof it is immaterial how many corners the extremaloid in question has.

46. Sufficient conditions for a minimum with respect to one-sided variations. If for the simple non-parametric problem stated in Chapter I, page 4, the class $[C_{12}]$ of arcs in which a minimizing arc is sought is restricted to admissible arcs which lie in a region R bounded by a curve B, then a minimizing arc may be made up in part of an arc of the boundary curve B. Necessary conditions for a minimum along such



arcs have been deduced by Weierstrass [1,p. 306] in his lectures and by Kneser [3,p.188] in 1900. In 1904 Bliss [107] proved the following sufficiency theorem for the minimum of the integral $\int_{x_1}^{x_2} f(x, y, y')\,dx$.

SUFFICIENCY THEOREM OF BLISS. If the arc E_{1342} is composed of two extremal arcs E_{13} and E_{42} and part of the boundary curve B_{34} ; if the problem is regular in R , Chapter II, page 19; if the arc E_{13} contains no point conjugate to 1 and the arc E_{34} contains no point conjugate to 4; if the extremal arcs E_{13} and E_{34} are tangent to B at the points 3 and 4 respectively; and if along B the inequality

$$f_y - \frac{d}{dx} f_{y'} < 0$$

holds; then the arc E_{1342} furnishes a strong relative minimum to the integral of the problem.

The method of proof is to construct a field composed of three parts, F_1 , F_2 , and F_3 . The fields F_1 and F_2 are constructed in the usual way about the extremal arcs E_{13} and E_{34} . The field F_3 is formed by a one parameter family of extremals one of which is tangent to the arc B_{34} at each point of that arc. Bliss proved that the fundamental sufficiency theorem with some modification holds in such a composite field.

Bolza [6, p.400] in 1909 established a similar result for the parametric problem. Likewise Hadamard [7, p.453] treated this case. In 1914 Bliss and Underhill [108] extended the sufficiency theorem to hold for space problems.

47. Closed extremals. The problem considered in this paragraph is that of finding in a class [C] of closed curves of the form

(3) $$x = x(t), \quad y = y(t) \qquad (-\infty < t < +\infty)$$

one which gives a minimum value to the integral

(4) $$I = \int_t^{t+1} F(x, y, x', y')\,dt .$$

The functions $x(t)$ and $y(t)$ satisfy the periodicity conditions

$$x(t+1) \equiv x(t), \quad y(t+1) \equiv y(t)$$

identically in t.

Such a problem was studied by Poincaré [109] in 1899 where $F(x, y, x', y') =$ constant was the vis viva integral for a dynamical problem in the plane. He sought to find necessary conditions and sufficient conditions for a minimum. In order to show that a closed extremal with equations of the form (3) affords a minimum to the integral (4), he set up a family of solutions of the Euler equations asymptotic to the closed extremal (3). Using the asymptotic solutions instead of the extremals of a field he made a proof that the closed extremal (3) furnishes a relative minimum for the problem.

Hadamard [7, p.432] treated the problem in a somewhat different way. He transformed the coordinate system into a

system of curvilinear coordinates (X, Y) in such a way that an extremal E transforms into the X-axis. A general closed curve passing through a point $P_1(x_1, y_1)$ transforms into an arc defined on the interval $(0, 1)$ with equal ordinates at the ends of the interval. This arc is repeated for every circuit of the curve. For the transformed problem in X, Y-space Hadamard set up a field of extremal arcs on the interval $(0, 1)$ each having equal ordinates at the ends. He stated the following

SUFFICIENCY THEOREM OF HADAMARD. If the extremal arc $Y = 0 \quad (0 \leqq X \leqq 1)$ is imbedded in a field of the kind described above; if this arc contains no point conjugate to $(0, 0)$; and if the condition II'_b of Weierstrass holds at every point of the field; then the closed extremal E furnishes a minimum with respect to all closed curves in a sufficiently small neighborhood of E.

Radon [110] in 1922 also proved this theorem and extended it to an n-dimensional problem.

Carathéodory [111] transformed the problem in the same way that Hadamard did and investigated for a minimum the transformed integral whose integrand function is periodic in t with period unity. He employed a different type of field from that of Hadamard in that all the extremals of his field pass through a point and are not periodic. He found sufficient conditions for a strong relative minimum and for a weak relative minimum for an n-space problem.

CHAPTER IX

MULTIPLE INTEGRALS

48. Introduction. A problem of the calculus of variations involving a double integral may be stated as follows: In a class of admissible surfaces $[S]$ of the form

(1) $$z = z(x, y)$$

where the function $z(x, y)$ is defined for all points of a bounded, closed region A, and where every surface passes through a given simply closed boundary curve L, to find a surface which gives a minimum value to the integral

(2) $$I(S) = \iint_A f(x, y, z, p, q)\, dx$$

where $p = z_x(x, y)$ and $q = z_y(x, y)$.

For the problem with a fixed boundary curve L the theory of necessary conditions and the Weierstrass fundamental formula are analogous to the corresponding theory for the simplest problem in the plane involving a simple integral. The necessary conditions of Euler [6, p.655] states that the function defining the minimizing surface must satisfy the equation

(3) $$f_z - \frac{\partial}{\partial x} f_p - \frac{\partial}{\partial y} f_q = 0$$

The problem also has analogues of the Legendre [6, p.673] and Jacobi [6, p.675] conditions.

The notion of a field can be extended to the double integral problem. In fact, the first publication of the field theory developed this idea for a double integral problem, as was said in Chapter II, introduction. This was in a paper of

Schwarz [112] on surfaces of minimum area, published in 1885. The next writer who knew the Weierstrass field theory was G. Kobb[113] who published in 1891 a paper on the minima of double integrals in parametric form. However, Kneser [114] pointed out that Kobb's reasoning in the proof of sufficient conditions for a strong relative minimum was inaccurate. Kobb stated an analogue of the fundamental formula of Weierstrass with conditions sufficient for a strong relative minimum using the

function. His reasoning justified, however, only the assertion that the extremal surface furnishes a weak relative minimum. Kneser's Lehrbuch [3, Abs.4] published in 1900 contained a proof of the sufficiency of a certain set of conditions for a weak relative minimum by an expansion method. Hilbert 115 extended his invariant integral theory to the double integral problem and gave in 1900 the first proof of the analogue of the fundamental formula of Weierstrass for this problem. Osgood [116] in 1902 and Bolza [6, pp.652-683] in 1909 have given accounts of this theory. Kneser [10,pp.328-335] proved a fundamental sufficiency theorem for a similar problem in parametric representation in 1925.

For regular problems Lichtenstein [120] in 1917 studied a boundary value problem associated with the analogue of the Jacobi condition for multiple integrals. He showed that Jacobi's necessary condition is equivalent to requiring that $\lambda_1 \geqq 1$ where λ_1 is the smallest positive characteristic number of the boundary value problem. If the Jacobi condition is satisfied for the extremal surface in the stronger form

$\lambda_1 > 1$ he proved that the surface can be imbedded in a field of extremal surfaces. With suitable further hypotheses one may deduce sufficient conditions for a strong relative minimum from this result by the method of Weierstrass as adapted to multiple integrals by Hilbert.

Not much has been accomplished in the way of sufficiency proofs for more general problems involving multiple integrals. Fields and sufficiency theorems for more general problems have been treated by Hilbert [115] in 1905, Hadamard [118] in 1905, Radon [119] in 1911, and Carathéodory [121] in 1929.

49. Fields and a fundamental sufficiency theorem for double integrals. In 1885 Schwarz [112] sought to prove that the portion of a minimal surface bounded by a given closed curve L has a smaller area than every other neighboring portion of a surface bounded by the same curve L. This is a problem of the calculus of variations involving a double integral. He introduced the notion of a one-parameter family of minimal surfaces, containing the particular minimal surface for a definite parameter value, and simply covering a region of x,y,z-space. By means of this family of minimal surfaces, he made the proof that in a region simply covered by such a family the minimal surface has actually a smaller area than every other continuous surface made up of portions of analytic surfaces and bounded by L.

The use of such one-parameter families in the general theory gives rise to the definition of a field. Given a family of extremal surface E_a of the form

(4) $$z = z(x, y, a)$$

then provided that $z_a(x, y, a_0) \neq 0$ on the particular extremal surface E_0, corresponding to $a = 0$, the equation (4) may be solved for a in terms of x, y, and z in a neighborhood of the points of E_0 and the solutions may be represented in the form

(5) $$a = a(x, y, z)$$

A region of x, y, z-space thus simply covered by such a family of extremal surfaces will be called a field. In 1900 Hilbert [115] defined the slope functions $p(x, y, z)$ and $q(x, y, z)$ of the field by the equations

(6) $$p(x, y, z) = z_x(x, y, a), \quad q(x, y, z) = z_y(x, y, a)$$

where the function (5) has been substituted for the parameter a. He showed that in a field the integral

(7) $$I^* = \iint \left\{ f + (z_x - p) f_p + (z_y - q) f_q \right\} dx\, dy$$

depends only on the contour which bounds the surface over which it is calculated.

The independence property of the integral (7) formed with the slope function (6) justifies the formula

(8) $$I(S) - I(E) = \iint_A E(x, y, z, p(x, y, z), q(x, y, z), z_x, z_y)\, dx\, dy$$

where E is an arbitrarily selected portion of an extremal surface of the field with boundary L and S is an arbitrary admissible surface through L. The E-function is defined as follows:

$$E(x, y, z, p, q, z_x, z_y) = f(x, y, z, z_x, z_y) - f(x, y, z, p, q) - (z_x - p) f_p(x, y, z, p, q) - (z_y - q) f_q(x, y, z, p, q)$$

and in the integrand the values x, y, z, z_x, z_y are those belonging to S. Formula (8) is a generalization of the fundamental formula of Weierstrass for simple integrals. It was a special case of this formula which Schwarz used to prove the least area property of minimal surfaces. The formula (8) enabled Hilbert to prove the fundamental sufficiency theorem in a field.

FUNDAMENTAL SUFFICIENCY THEOREM. *If a portion* E *of an extremal surface bounded by a curve* L *imbedded in a field* F, *and if the condition*

$$E(x, y, z, p(x, y, z), q(x, y, z), z_x, z_y) > 0$$

is satisfied for all admissible sets $(x, y, z, z_x, z_y) \neq (x, y, z, p, q)$ *for which the point* (x, y, z) *belongs to the field* F, *then* E *gives to the integral* (2) *a value smaller than that given by every other admissible surface* S *in* F *bounded by the same curve* L.

This statement of the theorem is essentially that of Bolza [6, p.683].

50. *Sufficient conditions for a relative minimum.* For the double integral problem, sufficient conditions for a strong relative minimum have been deduced for *regular problems*. A problem is said to be regular in a region of x, y, z-space if

the inequality

$$f_{pp}f_{qq} - f_{pq}^2 > 0$$

holds for every point of the region and for every p and q such that (x, y, z, p, q) constitutes an admissible set.

Lichenstein [120] showed in 1917 that for a regular problem a portion E of an extremal surface on which the analogue of the Jacobi condition in strengthened form is satisfied can be imbedded in a field. If these conditions, together with the strengthened form of the analogue of the Legendre condition hold, then E furnishes at least a strong relative minimum to the integral (2).

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In the following list the books are referred to by the numbers in square brackets in the text. The first group is composed of a few general works on the calculus of variations which are referred to occasionally throughout the dissertation. Thereafter the works are grouped according to the chapters in which they are described. This arrangement classifies the papers according to the methods of proof and also to a great extent according to their subject matter. Only the results of Chapters I and II are duplicated in later chapters. In each division of the bibliography the arrangement of the papers is chronological. The papers after the first chapter which are preceded by an asterisk in the bibliography do not treat sufficient conditions.

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VITA

William Larkin Duren, Jr. was born in Macon, Mississippi on November 10, 1905. He is the son of Reverend W. L. Duren and Mrs. Ethel Bennett Duren. He attended high school in Columbus, Mississippi, and later in New Orleans, Louisiana, where he graduated from Warren Easton Boys' High School in 1922. He entered Tulane University the same year and received the A. B. degree there in 1926. During his Senior year he assisted in the mathematics department and taught one class. The two years 1926-1928 he spent as an assistand and instructor in mathematics at Tulane and, at the same time, took work towards the A. M. degree which he received in 1928. He studied at that institution principally under Professor H. E. Buchanan and wrote a master's thesis in celestial mechanics.

He attended the University of Chicago for the first time in the Summer Quarter of 1926. He was a Fellow in mathematics there during the years 1928-1929 and 1929-1930. At the University of Chicago he studied under Professors Bliss, Graves, McMillan, Lane, Dickson, Barnard, Jackson, Bartky, Weaver, McDuffee, and Lunn; chiefly under Professor Bliss in the calculus of variations. To Professor Bliss he owes especial thanks for his patient and thought provoking supervision of the foregoing dissertation. Professor Buchanan guided the author from his Freshman year and was largely responsible for his coming to the University of Chicago to study. The author wishes to thank him as well as Professor Bliss.